6 □ APPLICATIONS OF INTEGRATION

6.1 Areas between Curves

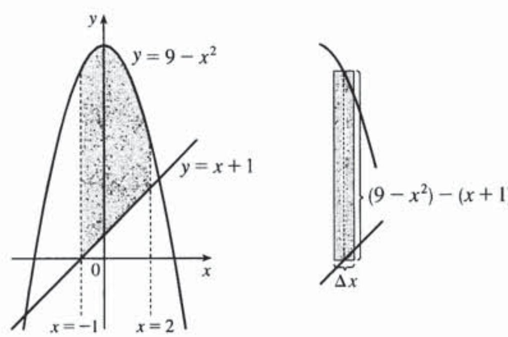
1. $A = \int_{x=0}^{x=4} (y_T - y_B)\,dx = \int_0^4 \left[(5x - x^2) - x\right] dx = \int_0^4 (4x - x^2)\,dx$
$= \left[2x^2 - \frac{1}{3}x^3\right]_0^4 = \left(32 - \frac{64}{3}\right) - (0) = \frac{32}{3}$

2. $A = \int_0^2 \left(\sqrt{x+2} - \frac{1}{x+1}\right) dx = \left[\frac{2}{3}(x+2)^{3/2} - \ln(x+1)\right]_0^2$
$= \left[\frac{2}{3}(4)^{3/2} - \ln 3\right] - \left[\frac{2}{3}(2)^{3/2} - \ln 1\right] = \frac{16}{3} - \ln 3 - \frac{4}{3}\sqrt{2}$

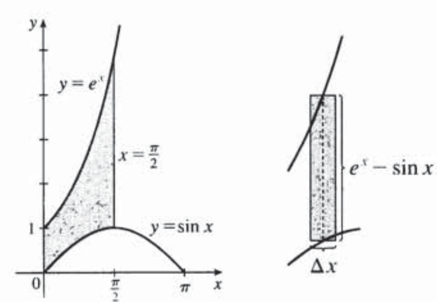
3. $A = \int_{y=-1}^{y=1} (x_R - x_L)\,dy = \int_{-1}^1 \left[e^y - (y^2 - 2)\right] dy$
$= \int_{-1}^1 (e^y - y^2 + 2)\,dy = \left[e^y - \frac{1}{3}y^3 + 2y\right]_{-1}^1 = \left(e^1 - \frac{1}{3} + 2\right) - \left(e^{-1} + \frac{1}{3} - 2\right) = e - \frac{1}{e} + \frac{10}{3}$

4. $A = \int_0^3 \left[(2y - y^2) - (y^2 - 4y)\right] dy = \int_0^3 (-2y^2 + 6y)\,dy$
$= \left[-\frac{2}{3}y^3 + 3y^2\right]_0^3 = (-18 + 27) - 0 = 9$

5. $A = \int_{-1}^2 \left[(9 - x^2) - (x+1)\right] dx$
$= \int_{-1}^2 (8 - x - x^2)\,dx$
$= \left[8x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2$
$= \left(16 - 2 - \frac{8}{3}\right) - \left(-8 - \frac{1}{2} + \frac{1}{3}\right)$
$= 22 - 3 + \frac{1}{2} = \frac{39}{2}$

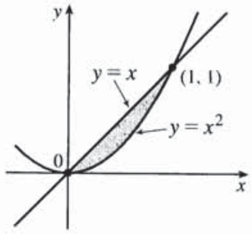


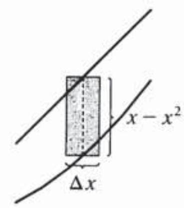
6. $A = \int_0^{\pi/2} (e^x - \sin x)\,dx$
$= \left[e^x + \cos x\right]_0^{\pi/2}$
$= \left(e^{\pi/2} + 0\right) - (1 + 1)$
$= e^{\pi/2} - 2$



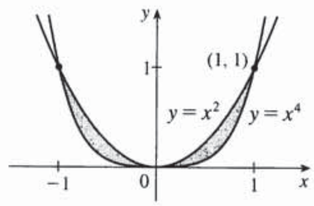
7. The curves intersect when $x = x^2 \Rightarrow x^2 - x = 0 \Leftrightarrow x(x-1) = 0 \Leftrightarrow x = 0, 1$.

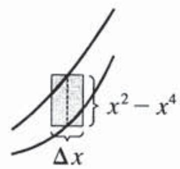
$$\begin{aligned} A &= \int_0^1 (x - x^2)\,dx \\ &= \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 \\ &= \tfrac{1}{2} - \tfrac{1}{3} \\ &= \tfrac{1}{6} \end{aligned}$$



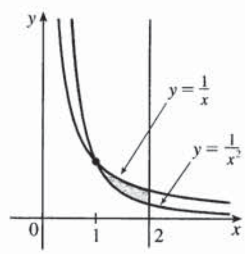


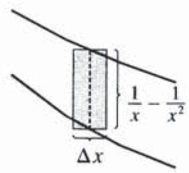
8. $$\begin{aligned} A &= \int_{-1}^1 (x^2 - x^4)\,dx \\ &= 2\int_0^1 (x^2 - x^4)\,dx \\ &= 2\left[\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right]_0^1 \\ &= 2\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = \tfrac{4}{15} \end{aligned}$$





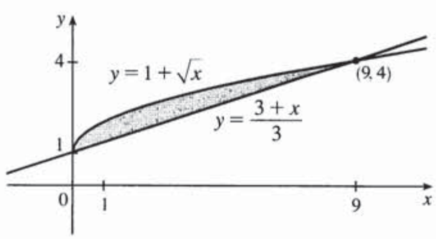
9. $$\begin{aligned} A &= \int_1^2 \left(\frac{1}{x} - \frac{1}{x^2}\right)dx = \left[\ln x + \frac{1}{x}\right]_1^2 \\ &= \left(\ln 2 + \tfrac{1}{2}\right) - (\ln 1 + 1) \\ &= \ln 2 - \tfrac{1}{2} \approx 0.19 \end{aligned}$$

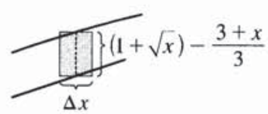




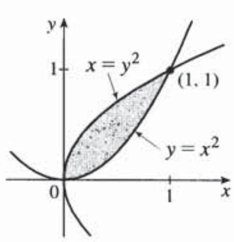
10. $1 + \sqrt{x} = \dfrac{3 + x}{3} = 1 + \dfrac{x}{3} \Rightarrow \sqrt{x} = \dfrac{x}{3} \Rightarrow x = \dfrac{x^2}{9} \Rightarrow 9x - x^2 = 0 \Rightarrow x(9 - x) = 0 \Rightarrow x = 0$ or 9, so

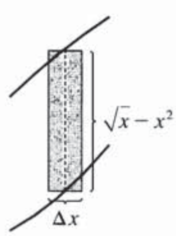
$$\begin{aligned} A &= \int_0^9 \left[(1 + \sqrt{x}) - \left(\frac{3 + x}{3}\right)\right]dx = \int_0^9 \left[(1 + \sqrt{x}) - \left(1 + \frac{x}{3}\right)\right]dx \\ &= \int_0^9 \left(\sqrt{x} - \tfrac{1}{3}x\right)dx = \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{6}x^2\right]_0^9 = 18 - \tfrac{27}{2} = \tfrac{9}{2} \end{aligned}$$





11. $$\begin{aligned} A &= \int_0^1 (\sqrt{x} - x^2)\,dx \\ &= \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{3}x^3\right]_0^1 \\ &= \tfrac{2}{3} - \tfrac{1}{3} \\ &= \tfrac{1}{3} \end{aligned}$$

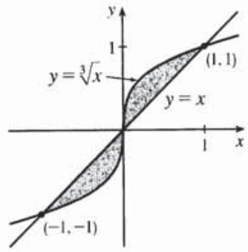


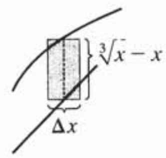


12. $x = \sqrt[3]{x} \;\Rightarrow\; x^3 = x \;\Rightarrow\; x^3 - x = 0 \;\Rightarrow\; x(x^2 - 1) = 0 \;\Rightarrow\; x(x+1)(x-1) = 0 \;\Rightarrow$
$x = -1, 0$, or 1, so

$$A = \int_{-1}^{1} \left| \sqrt[3]{x} - x \right| dx = \int_{-1}^{0} \left(x - \sqrt[3]{x} \right) dx + \int_{0}^{1} \left(\sqrt[3]{x} - x \right) dx = 2 \int_{0}^{1} \left(x^{1/3} - x \right) dx \qquad \text{[by symmetry]}$$

$$= 2\left[\tfrac{3}{4} x^{4/3} - \tfrac{1}{2} x^2 \right]_0^1 = 2\left(\tfrac{3}{4} - \tfrac{1}{2} \right) = \tfrac{1}{2}$$

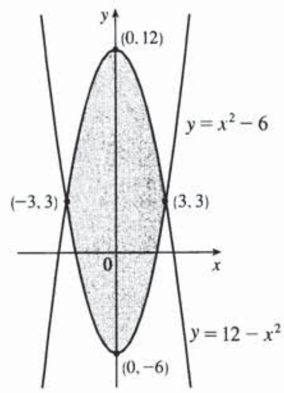


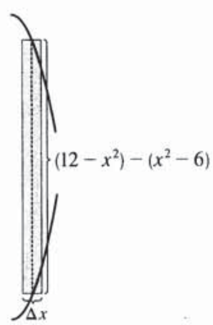


13. $12 - x^2 = x^2 - 6 \;\Leftrightarrow\; 2x^2 = 18 \;\Leftrightarrow\; x^2 = 9 \;\Leftrightarrow\; x^2 = \pm 3$, so

$$A = \int_{-3}^{3} \left[(12 - x^2) - (x^2 - 6) \right] dx = 2 \int_{0}^{3} (18 - 2x^2)\, dx \quad \text{[by symmetry]}$$

$$= 2\left[18x - \tfrac{2}{3} x^3 \right]_0^3 = 2\left[(54 - 18) - 0 \right] = 2(36) = 72$$

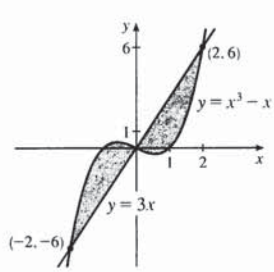


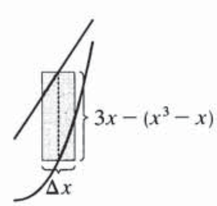


14. $x^3 - x = 3x \;\Rightarrow\; x^3 - 4x = 0 \;\Rightarrow\; x(x^2 - 4) = 0 \;\Rightarrow\; x(x+2)(x-2) = 0 \;\Rightarrow\; x = 0, -2$, or 2.
By symmetry,

$$A = \int_{-2}^{2} \left| 3x - (x^3 - x) \right| dx = 2 \int_{0}^{2} \left[3x - (x^3 - x) \right] dx = 2 \int_{0}^{2} (4x - x^3)\, dx = 2\left[2x^2 - \tfrac{1}{4} x^4 \right]_0^2$$

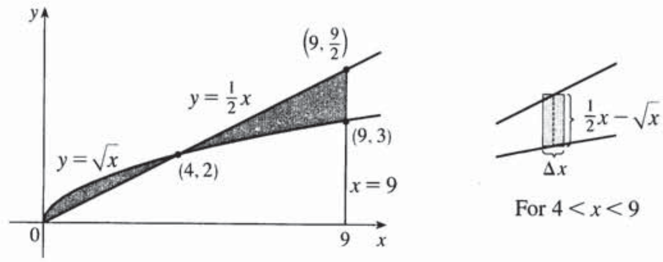
$$= 2(8 - 4) = 8$$





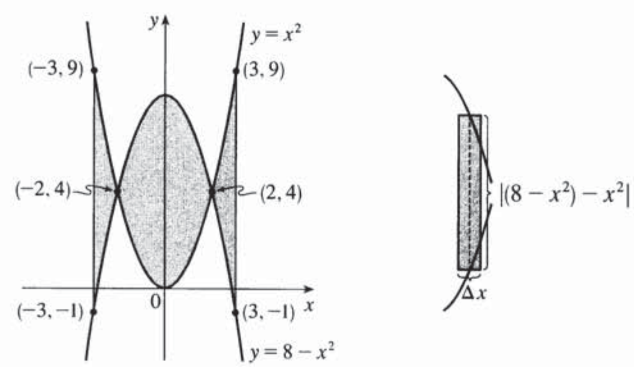
15. $\frac{1}{2}x = \sqrt{x} \;\Rightarrow\; \frac{1}{4}x^2 = x \;\Rightarrow\; x^2 - 4x = 0 \;\Rightarrow\; x(x-4) = 0 \;\Rightarrow\; x = 0$ or 4, so

$$A = \int_0^4 \left(\sqrt{x} - \tfrac{1}{2}x\right) dx + \int_4^9 \left(\tfrac{1}{2}x - \sqrt{x}\right) dx = \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{4}x^2\right]_0^4 + \left[\tfrac{1}{4}x^2 - \tfrac{2}{3}x^{3/2}\right]_4^9$$
$$= \left[\left(\tfrac{16}{3} - 4\right) - 0\right] + \left[\left(\tfrac{81}{4} - 18\right) - \left(4 - \tfrac{16}{3}\right)\right] = \tfrac{81}{4} + \tfrac{32}{3} - 26 = \tfrac{59}{12}$$



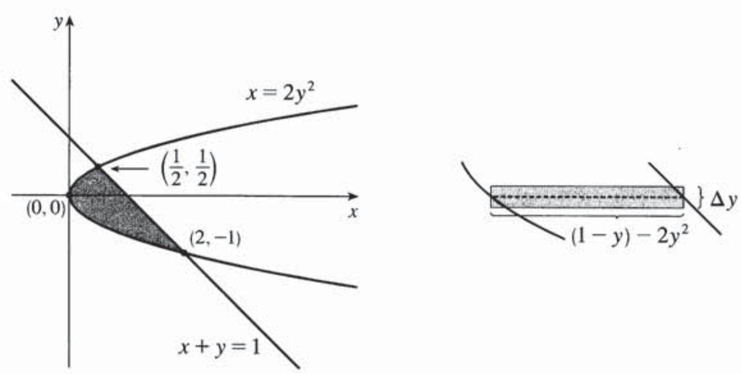
16. $A = \int_{-3}^{3} \left|(8 - x^2) - x^2\right| dx = 2\int_0^3 \left|8 - 2x^2\right| dx = 2\int_0^2 (8 - 2x^2)\, dx + 2\int_2^3 (2x^2 - 8)\, dx$

$$= 2\left[8x - \tfrac{2}{3}x^3\right]_0^2 + 2\left[\tfrac{2}{3}x^3 - 8x\right]_2^3 = 2\left[\left(16 - \tfrac{16}{3}\right) - 0\right] + 2\left[(18 - 24) - \left(\tfrac{16}{3} - 16\right)\right]$$
$$= 32 - \tfrac{32}{3} + 20 - \tfrac{32}{3} = 52 - \tfrac{64}{3} = \tfrac{92}{3}$$



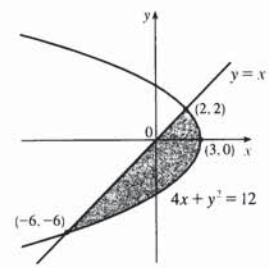
17. $2y^2 = 1 - y \;\Leftrightarrow\; 2y^2 + y - 1 = 0 \;\Leftrightarrow\; (2y-1)(y+1) = 0 \;\Leftrightarrow\; y = \frac{1}{2}$ or -1, so $x = \frac{1}{2}$ or 2 and

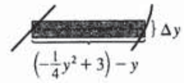
$$A = \int_{-1}^{1/2} \left[(1 - y) - 2y^2\right] dy = \int_{-1}^{1/2} \left(1 - y - 2y^2\right) dy = \left[y - \tfrac{1}{2}y^2 - \tfrac{2}{3}y^3\right]_{-1}^{1/2}$$
$$= \left(\tfrac{1}{2} - \tfrac{1}{8} - \tfrac{1}{12}\right) - \left(-1 - \tfrac{1}{2} + \tfrac{2}{3}\right) = \tfrac{7}{24} - \left(-\tfrac{5}{6}\right) = \tfrac{7}{24} + \tfrac{20}{24} = \tfrac{27}{24} = \tfrac{9}{8}$$



18. $4x + x^2 = 12 \quad \Leftrightarrow \quad (x+6)(x-2) = 0 \quad \Leftrightarrow \quad x = -6$ or $x = 2$, so $y = -6$ or $y = 2$ and

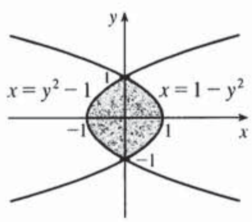
$$A = \int_{-6}^{2} \left[\left(-\tfrac{1}{4}y^2 + 3\right) - y\right] dy = \left[-\tfrac{1}{12}y^3 - \tfrac{1}{2}y^2 + 3y\right]_{-6}^{2} = \left(-\tfrac{2}{3} - 2 + 6\right) - (18 - 18 - 18) = 22 - \tfrac{2}{3} = \tfrac{64}{3}.$$

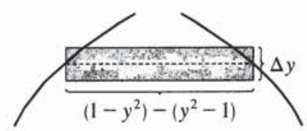




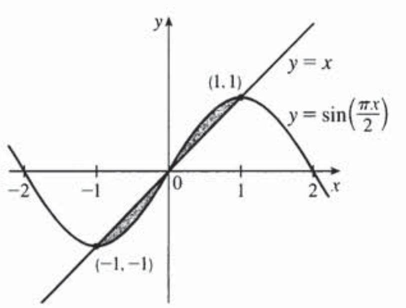
19. The curves intersect when $1 - y^2 = y^2 - 1 \quad \Leftrightarrow \quad 2 = 2y^2 \quad \Leftrightarrow \quad y^2 = 1 \quad \Leftrightarrow \quad y = \pm 1$.

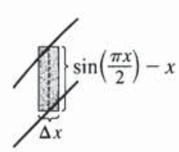
$$\begin{aligned} A &= \int_{-1}^{1} \left[(1 - y^2) - (y^2 - 1)\right] dy \\ &= \int_{-1}^{1} 2(1 - y^2)\, dy \\ &= 2 \cdot 2 \int_{0}^{1} (1 - y^2)\, dy \\ &= 4\left[y - \tfrac{1}{3}y^3\right]_0^1 = 4\left(1 - \tfrac{1}{3}\right) = \tfrac{8}{3} \end{aligned}$$





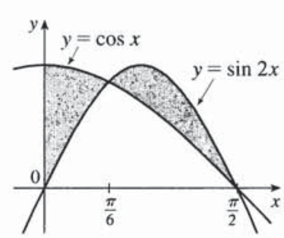
20. $A = 2\int_0^1 \left[\sin\left(\frac{\pi x}{2}\right) - x\right] dx = 2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{x^2}{2}\right]_0^1 = 2\left[\left(0 - \frac{1}{2}\right) - \left(-\frac{2}{\pi} - 0\right)\right] = \frac{4}{\pi} - 1$



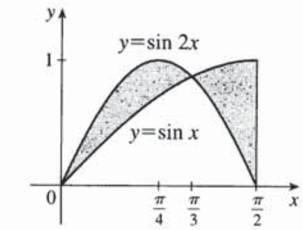


21. Notice that $\cos x = \sin 2x = 2\sin x \cos x \quad \Leftrightarrow$
$2\sin x \cos x - \cos x = 0 \quad \Leftrightarrow \quad \cos x\,(2\sin x - 1) = 0 \quad \Leftrightarrow$
$2\sin x = 1$ or $\cos x = 0 \quad \Leftrightarrow \quad x = \frac{\pi}{6}$ or $\frac{\pi}{2}$.

$$\begin{aligned} A &= \int_0^{\pi/6} (\cos x - \sin 2x)\, dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\, dx \\ &= \left[\sin x + \tfrac{1}{2}\cos 2x\right]_0^{\pi/6} + \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} - \left(0 + \tfrac{1}{2}\cdot 1\right) + \left(\tfrac{1}{2} - 1\right) - \left(-\tfrac{1}{2}\cdot\tfrac{1}{2} - \tfrac{1}{2}\right) = \tfrac{1}{2} \end{aligned}$$



22. $\sin x = \sin 2x = 2\sin x \cos x$ when $\sin x = 0$ and when $\cos x = \frac{1}{2}$; that is, when $x = 0$ or $\frac{\pi}{3}$.



$$A = \int_0^{\pi/3} (\sin 2x - \sin x)\, dx + \int_{\pi/3}^{\pi/2} (\sin x - \sin 2x)\, dx$$

$$= \left[-\tfrac{1}{2}\cos 2x + \cos x\right]_0^{\pi/3} + \left[\tfrac{1}{2}\cos 2x - \cos x\right]_{\pi/3}^{\pi/2}$$

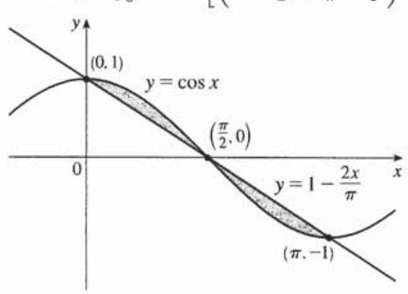
$$= \left[-\tfrac{1}{2}\left(-\tfrac{1}{2}\right) + \tfrac{1}{2}\right] - \left(-\tfrac{1}{2} + 1\right)$$

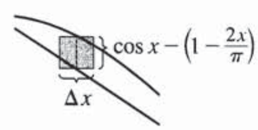
$$+ \left(-\tfrac{1}{2} - 0\right) - \left[\tfrac{1}{2}\left(-\tfrac{1}{2}\right) - \tfrac{1}{2}\right] = \tfrac{1}{2}$$

23. From the graph, we see that the curves intersect at $x = 0$, $x = \frac{\pi}{2}$, and $x = \pi$. By symmetry,

$$A = \int_0^{\pi} \left|\cos x - \left(1 - \frac{2x}{\pi}\right)\right| dx = 2\int_0^{\pi/2} \left[\cos x - \left(1 - \frac{2x}{\pi}\right)\right] dx = 2\int_0^{\pi/2} \left(\cos x - 1 + \frac{2x}{\pi}\right) dx$$

$$= 2\left[\sin x - x + \tfrac{1}{\pi}x^2\right]_0^{\pi/2} = 2\left[\left(1 - \tfrac{\pi}{2} + \tfrac{1}{\pi}\cdot\tfrac{\pi^2}{4}\right) - 0\right] = 2\left(1 - \tfrac{\pi}{2} + \tfrac{\pi}{4}\right) = 2 - \tfrac{\pi}{2}$$

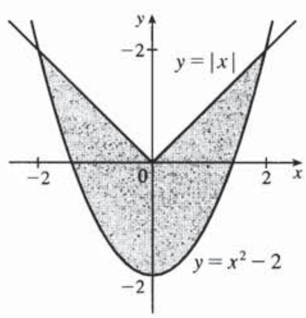


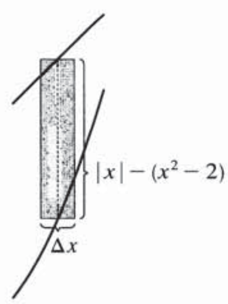


24. For $x > 0$, $x = x^2 - 2 \;\Rightarrow\; 0 = x^2 - x - 2 \;\Rightarrow\; 0 = (x-2)(x+1) \;\Rightarrow\; x = 2$. By symmetry,

$$\int_{-2}^{2} \left[|x| - (x^2 - 2)\right] dx = 2\int_0^2 \left[x - (x^2 - 2)\right] dx = 2\int_0^2 (x - x^2 + 2)\,dx = 2\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + 2x\right]_0^2$$

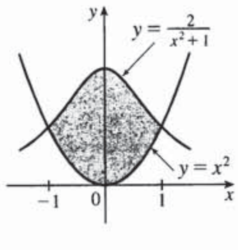
$$= 2\left(2 - \tfrac{8}{3} + 4\right) = \tfrac{20}{3}$$

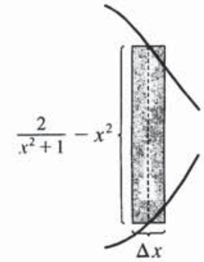




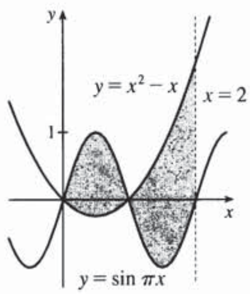
25. The curves intersect when $x^2 = \dfrac{2}{x^2 + 1} \;\Leftrightarrow\; x^4 + x^2 = 2 \;\Leftrightarrow\; x^4 + x^2 - 2 = 0 \;\Leftrightarrow\; (x^2 + 2)(x^2 - 1) = 0 \;\Leftrightarrow\; x^2 = 1 \;\Leftrightarrow\; x = \pm 1$.

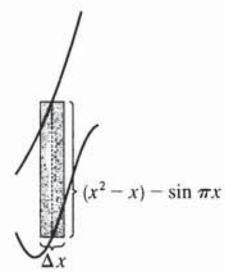
$$\begin{aligned} A &= \int_{-1}^{1}\left(\frac{2}{x^2+1} - x^2\right)dx \\ &= 2\int_{0}^{1}\left(\frac{2}{x^2+1} - x^2\right)dx \\ &= 2\left[2\tan^{-1}x - \tfrac{1}{3}x^3\right]_0^1 = 2\left(2\cdot\tfrac{\pi}{4} - \tfrac{1}{3}\right) \\ &= \pi - \tfrac{2}{3} \approx 2.47 \end{aligned}$$





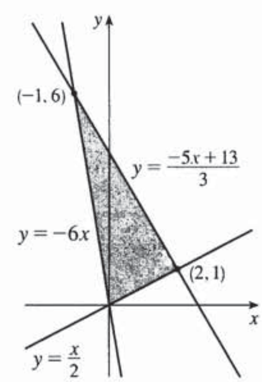
26. $$\begin{aligned} A &= \int_0^1 \left[\sin \pi x - (x^2 - x)\right] dx + \int_1^2 \left[(x^2 - x) - \sin \pi x\right] dx \\ &= \left[-\tfrac{1}{\pi}\cos \pi x - \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2\right]_0^1 + \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{\pi}\cos \pi x\right]_1^2 \\ &= \left(\tfrac{1}{\pi} - \tfrac{1}{3} + \tfrac{1}{2}\right) - \left(-\tfrac{1}{\pi}\right) + \left(\tfrac{8}{3} - 2 + \tfrac{1}{\pi}\right) - \left(\tfrac{1}{3} - \tfrac{1}{2} - \tfrac{1}{\pi}\right) \\ &= \tfrac{4}{\pi} + 1 \end{aligned}$$



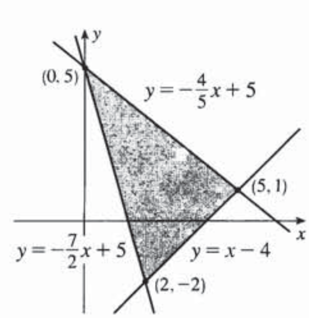


27. An equation of the line through $(0, 0)$ and $(2, 1)$ is $y = \frac{1}{2}x$; through $(0, 0)$ and $(-1, 6)$ is $y = -6x$; through $(2, 1)$ and $(-1, 6)$ is $y = -\frac{5}{3}x + \frac{13}{3}$.

$$\begin{aligned} A &= \int_{-1}^{0} \left[\left(-\tfrac{5}{3}x + \tfrac{13}{3}\right) - (-6x)\right] dx + \int_0^2 \left[\left(-\tfrac{5}{3}x + \tfrac{13}{3}\right) - \tfrac{1}{2}x\right] dx \\ &= \int_{-1}^{0} \left(\tfrac{13}{3}x + \tfrac{13}{3}\right) dx + \int_0^2 \left(-\tfrac{13}{6}x + \tfrac{13}{3}\right) dx \\ &= \tfrac{13}{3}\int_{-1}^{0} (x+1)\, dx + \tfrac{13}{3}\int_0^2 \left(-\tfrac{1}{2}x + 1\right) dx \\ &= \tfrac{13}{3}\left[\tfrac{1}{2}x^2 + x\right]_{-1}^{0} + \tfrac{13}{3}\left[-\tfrac{1}{4}x^2 + x\right]_0^2 \\ &= \tfrac{13}{3}\left[0 - \left(\tfrac{1}{2} - 1\right)\right] + \tfrac{13}{3}[(-1+2) - 0] = \tfrac{13}{3}\cdot\tfrac{1}{2} + \tfrac{13}{3}\cdot 1 = \tfrac{13}{2} \end{aligned}$$



28. $$\begin{aligned} A &= \int_0^2 \left[\left(-\tfrac{4}{5}x + 5\right) - \left(-\tfrac{7}{2}x + 5\right)\right] dx + \int_2^5 \left[\left(-\tfrac{4}{5}x + 5\right) - (x - 4)\right] dx \\ &= \int_0^2 \tfrac{27}{10}x\, dx + \int_2^5 \left(-\tfrac{9}{5}x + 9\right) dx \\ &= \left[\tfrac{27}{20}x^2\right]_0^2 + \left[-\tfrac{9}{10}x^2 + 9x\right]_2^5 \\ &= \left(\tfrac{27}{5} - 0\right) + \left(-\tfrac{45}{2} + 45\right) - \left(-\tfrac{18}{5} + 18\right) = \tfrac{27}{2} \end{aligned}$$

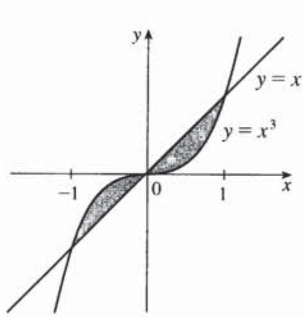


29. $A = \int_{-1}^{1} \left|x^3 - x\right| dx$

$= 2\int_0^1 \left(x - x^3\right) dx$ [by symmetry]

$= 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1$

$= 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$



30. The curves intersect when $\sqrt{x+2} = x \;\Rightarrow\; x + 2 = x^2 \;\Rightarrow\; x^2 - x - 2 = 0 \;\Rightarrow$ $(x-2)(x+1) = 0 \;\Rightarrow\; x = -1$ or 2. [-1 is extraneous]

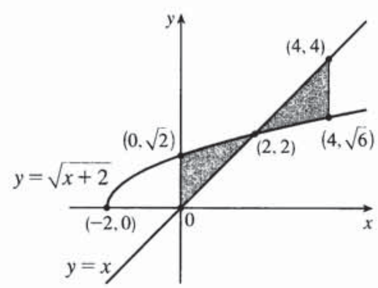
$A = \int_0^4 \left|\sqrt{x+2} - x\right| dx$

$= \int_0^2 \left(\sqrt{x+2} - x\right) dx + \int_2^4 \left(x - \sqrt{x+2}\right) dx$

$= \left[\frac{2}{3}(x+2)^{3/2} - \frac{1}{2}x^2\right]_0^2 + \left[\frac{1}{2}x^2 - \frac{2}{3}(x+2)^{3/2}\right]_2^4$

$= \left(\frac{16}{3} - 2\right) - \left(\frac{2}{3}\left(2\sqrt{2}\right) - 0\right) + \left(8 - \frac{2}{3}\left(6\sqrt{6}\right)\right) - \left(2 - \frac{16}{3}\right)$

$= 4 + \frac{32}{3} - \frac{4}{3}\sqrt{2} - 4\sqrt{6} = \frac{44}{3} - 4\sqrt{6} - \frac{4}{3}\sqrt{2}$



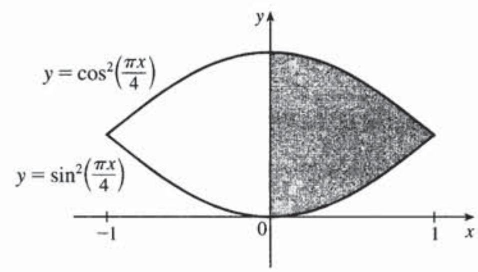
31. Let $f(x) = \cos^2\left(\frac{\pi x}{4}\right) - \sin^2\left(\frac{\pi x}{4}\right)$ and $\Delta x = \frac{1-0}{4}$.

The shaded area is given by

$A = \int_0^1 f(x)\,dx \approx M_4$

$= \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right]$

≈ 0.6407



32. The curves intersect when $\sqrt[3]{16 - x^3} = x \;\Rightarrow$

$16 - x^3 = x^3 \;\Rightarrow\; 2x^3 = 16 \;\Rightarrow\; x^3 = 8 \;\Rightarrow\; x = 2.$

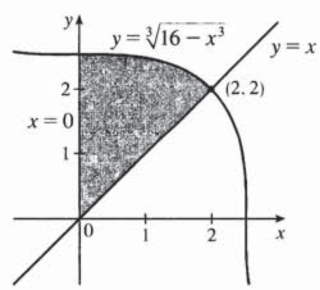
Let $f(x) = \sqrt[3]{16 - x^3} - x$ and $\Delta x = \frac{2-0}{4}$.

The shaded area is given by

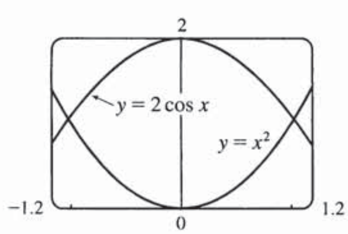
$A = \int_0^2 f(x)\,dx \approx M_4$

$= \frac{2}{4}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right)\right]$

≈ 2.8144



33.

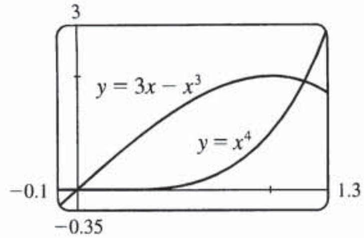


From the graph, we see that the curves intersect at $x = \pm a \approx \pm 1.02$, with $2\cos x > x^2$ on $(-a, a)$. So the area of the region bounded by the curves is

$A = \int_{-a}^{a} \left(2\cos x - x^2\right) dx = 2\int_0^a \left(2\cos x - x^2\right) dx$

$= 2\left[2\sin x - \frac{1}{3}x^3\right]_0^a \approx 2.70$

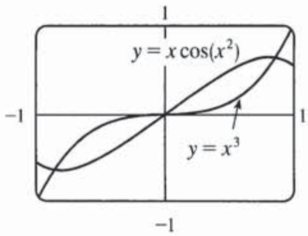
34.



From the graph, we see that the curves intersect at $x = 0$ and at $x = a \approx 1.17$, with $3x - x^3 > x^4$ on $(0, a)$. So the area of the region bounded by the curves is

$$A = \int_0^a \left[(3x - x^3) - x^4\right] dx = \left[\tfrac{3}{2}x^2 - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5\right]_0^a \approx 1.15$$

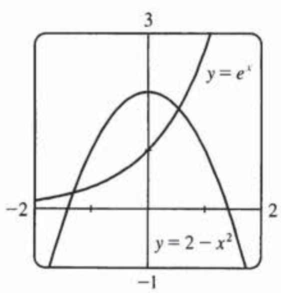
35.



From the graph, we see that the curves intersect at $x = \pm a \approx \pm 0.86$. So the area of the region bounded by the curves is

$$A = 2\int_0^a \left[x\cos(x^2) - x^3\right] dx = 2\left[\tfrac{1}{2}\sin(x^2) - \tfrac{1}{4}x^4\right]_0^a \approx 0.40$$

36.

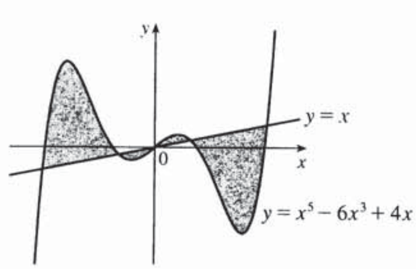


From the graph, we see that the curves intersect at $x = a \approx -1.32$ and $x = b \approx 0.54$, with $2 - x^2 > e^x$ on (a, b). So the area of the region bounded by the curves is

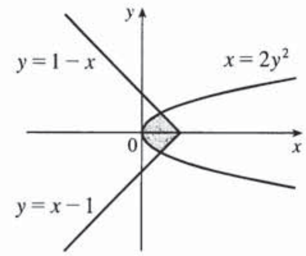
$$A = \int_a^b \left[(2 - x^2) - e^x\right] dx = \left[2x - \tfrac{1}{3}x^3 - e^x\right]_a^b \approx 1.45$$

37. As the figure illustrates, the curves $y = x$ and $y = x^5 - 6x^3 + 4x$ enclose a four-part region symmetric about the origin (since $x^5 - 6x^3 + 4x$ and x are odd functions of x). The curves intersect at values of x where $x^5 - 6x^3 + 4x = x$; that is, where $x(x^4 - 6x^2 + 3) = 0$. That happens at $x = 0$ and where $x^2 = \dfrac{6 \pm \sqrt{36 - 12}}{2} = 3 \pm \sqrt{6}$; that is, at $x = -\sqrt{3 + \sqrt{6}}, -\sqrt{3 - \sqrt{6}}, 0, \sqrt{3 - \sqrt{6}}$, and $\sqrt{3 + \sqrt{6}}$. The exact area is

$$\begin{aligned}2\int_0^{\sqrt{3+\sqrt{6}}} \left|(x^5 - 6x^3 + 4x) - x\right| dx &= 2\int_0^{\sqrt{3+\sqrt{6}}} \left|x^5 - 6x^3 + 3x\right| dx \\ &= 2\int_0^{\sqrt{3-\sqrt{6}}} (x^5 - 6x^3 + 3x)\, dx + 2\int_{\sqrt{3-\sqrt{6}}}^{\sqrt{3+\sqrt{6}}} (-x^5 + 6x^3 - 3x)\, dx \\ &\overset{\text{CAS}}{=} 12\sqrt{6} - 9\end{aligned}$$



38. The inequality $x \ge 2y^2$ describes the region that lies on, or to the right of, the parabola $x = 2y^2$. The inequality $x \le 1 - |y|$ describes the region that lies on, or to the left of, the curve $x = 1 - |y| = \begin{cases} 1 - y & \text{if } y \ge 0 \\ 1 + y & \text{if } y < 0 \end{cases}$.



So the given region is the shaded region that lies between the curves.

The graphs of $x = 1 - y$ and $x = 2y^2$ intersect when $1 - y = 2y^2 \;\Leftrightarrow$
$2y^2 + y - 1 = 0 \;\Leftrightarrow\; (2y - 1)(y + 1) = 0 \;\Rightarrow\; y = \frac{1}{2}$ (for $y \ge 0$).
By symmetry,

$$A = 2\int_0^{1/2} \left[(1 - y) - 2y^2\right] dy = 2\left[-\tfrac{2}{3}y^3 - \tfrac{1}{2}y^2 + y\right]_0^{1/2} = 2\left[\left(-\tfrac{1}{12} - \tfrac{1}{8} + \tfrac{1}{2}\right) - 0\right] = 2\left(\tfrac{7}{24}\right) = \tfrac{7}{12}.$$

39. 1 second $= \frac{1}{3600}$ hour, so 10 s $= \frac{1}{360}$ h. With the given data, we can take $n = 5$ to use the Midpoint Rule.
$\Delta t = \frac{1/360 - 0}{5} = \frac{1}{1800}$, so

$$\begin{aligned} \text{distance}_{\text{Kelly}} - \text{distance}_{\text{Chris}} &= \int_0^{1/360} v_K\,dt - \int_0^{1/360} v_C\,dt = \int_0^{1/360} (v_K - v_C)\,dt \\ &\approx M_5 = \tfrac{1}{1800}\left[(v_K - v_C)(1) + (v_K - v_C)(3) + (v_K - v_C)(5)\right. \\ &\qquad \left. + (v_K - v_C)(7) + (v_K - v_C)(9)\right] \\ &= \tfrac{1}{1800}\left[(22 - 20) + (52 - 46) + (71 - 62) + (86 - 75) + (98 - 86)\right] \\ &= \tfrac{1}{1800}(2 + 6 + 9 + 11 + 12) = \tfrac{1}{1800}(40) = \tfrac{1}{45} \text{ mile, or } 117\tfrac{1}{3} \text{ feet} \end{aligned}$$

40. If $x =$ distance from left end of pool and $w = w(x) =$ width at x, then the Midpoint Rule with $n = 4$ and
$\Delta x = \dfrac{b - a}{n} = \dfrac{8 \cdot 2 - 0}{4} = 4$ gives Area $= \int_0^{16} w\,dx \approx 4(6.2 + 6.8 + 5.0 + 4.8) = 4(22.8) = 91.2 \text{ m}^2$.

41. We know that the area under curve A between $t = 0$ and $t = x$ is $\int_0^x v_A(t)\,dt = s_A(x)$, where $v_A(t)$ is the velocity of car A and s_A is its displacement. Similarly, the area under curve B between $t = 0$ and $t = x$ is $\int_0^x v_B(t)\,dt = s_B(x)$.

(a) After one minute, the area under curve A is greater than the area under curve B. So car A is ahead after one minute.

(b) The area of the shaded region has numerical value $s_A(1) - s_B(1)$, which is the distance by which A is ahead of B after 1 minute.

(c) After two minutes, car B is traveling faster than car A and has gained some ground, but the area under curve A from $t = 0$ to $t = 2$ is still greater than the corresponding area for curve B, so car A is still ahead.

(d) From the graph, it appears that the area between curves A and B for $0 \le t \le 1$ (when car A is going faster), which corresponds to the distance by which car A is ahead, seems to be about 3 squares. Therefore, the cars will be side by side at the time x where the area between the curves for $1 \le t \le x$ (when car B is going faster) is the same as the area for $0 \le t \le 1$. From the graph, it appears that this time is $x \approx 2.2$. So the cars are side by side when $t \approx 2.2$ minutes.

42. The area under $R'(x)$ from $x = 50$ to $x = 100$ represents the change in revenue, and the area under $C'(x)$ from $x = 50$ to $x = 100$ represents the change in cost. The shaded region represents the difference between these two values; that is, the increase in profit as the production level increases from 50 units to 100 units. We use the