9.3 Separable Equations

1. $\dfrac{dy}{dx} = \dfrac{y}{x} \;\Rightarrow\; \dfrac{dy}{y} = \dfrac{dx}{x}\;\; [y \neq 0] \;\Rightarrow\; \displaystyle\int \frac{dy}{y} = \int \frac{dx}{x} \;\Rightarrow\; \ln|y| = \ln|x| + C \;\Rightarrow$

$|y| = e^{\ln|x|+C} = e^{\ln|x|}e^C = e^C\,|x| \;\Rightarrow\; y = Kx$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

2. $\dfrac{dy}{dx} = \dfrac{e^{2x}}{4y^3} \;\Rightarrow\; 4y^3\,dy = e^{2x}\,dx \;\Rightarrow\; \int 4y^3\,dy = \int e^{2x}\,dx \;\Rightarrow\; y^4 = \frac{1}{2}e^{2x} + C \;\Rightarrow$

$y = \pm\sqrt[4]{\frac{1}{2}e^{2x} + C}$

3. $(x^2+1)y' = xy \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{xy}{x^2+1} \;\Rightarrow\; \dfrac{dy}{y} = \dfrac{x\,dx}{x^2+1}\;\; [y \neq 0] \;\Rightarrow\; \displaystyle\int \frac{dy}{y} = \int \frac{x\,dx}{x^2+1} \;\Rightarrow$

$\ln|y| = \frac{1}{2}\ln(x^2+1) + C\;\; [u = x^2+1,\, du = 2x\,dx] \; = \ln(x^2+1)^{1/2} + \ln e^C = \ln\!\left(e^C\sqrt{x^2+1}\right) \;\Rightarrow$

$|y| = e^C\sqrt{x^2+1} \;\Rightarrow\; y = K\sqrt{x^2+1}$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

4. $y' = y^2\sin x \;\Rightarrow\; \dfrac{dy}{dx} = y^2\sin x \;\Rightarrow\; \dfrac{dy}{y^2} = \sin x\,dx\;\; [y \neq 0] \;\Rightarrow\; \displaystyle\int \frac{dy}{y^2} = \int \sin x\,dx \;\Rightarrow$

$-\dfrac{1}{y} = -\cos x + C \;\Rightarrow\; \dfrac{1}{y} = \cos x - C \;\Rightarrow\; y = \dfrac{1}{\cos x + K}$, where $K = -C$. $y = 0$ is also a solution.

5. $(1+\tan y)\,y' = x^2+1 \;\Rightarrow\; (1+\tan y)\dfrac{dy}{dx} = x^2+1 \;\Rightarrow\; \left(1 + \dfrac{\sin y}{\cos y}\right)dy = (x^2+1)\,dx \;\Rightarrow$

$\displaystyle\int\left(1 - \frac{-\sin y}{\cos y}\right)dy = \int (x^2+1)\,dx \;\Rightarrow\; y - \ln|\cos y| = \tfrac{1}{3}x^3 + x + C$. Note: The left side is equivalent to $y + \ln|\sec y|$.

6. $\dfrac{du}{dr} = \dfrac{1+\sqrt{r}}{1+\sqrt{u}} \;\Rightarrow\; (1+\sqrt{u})\,du = (1+\sqrt{r})\,dr \;\Rightarrow\; \int (1+u^{1/2})\,du = \int (1+r^{1/2})\,dr \;\Rightarrow$

$u + \frac{2}{3}u^{3/2} = r + \frac{2}{3}r^{3/2} + C$

7. $\dfrac{dy}{dt} = \dfrac{te^t}{y\sqrt{1+y^2}} \;\Rightarrow\; y\sqrt{1+y^2}\,dy = te^t\,dt \;\Rightarrow\; \int y\sqrt{1+y^2}\,dy = \int te^t\,dt \;\Rightarrow$

$\frac{1}{3}\left(1+y^2\right)^{3/2} = te^t - e^t + C$ [where the first integral is evaluated by substitution and the second by parts] $\Rightarrow$

$1 + y^2 = [3(te^t - e^t + C)]^{2/3} \;\Rightarrow\; y = \pm\sqrt{[3(te^t - e^t + C)]^{2/3} - 1}$

8. $y' = \dfrac{xy}{2\ln y} \;\Rightarrow\; \dfrac{2\ln y}{y}\,dy = x\,dx \;\Rightarrow\; \displaystyle\int \frac{2\ln y}{y}\,dy = \int x\,dx \;\Rightarrow\; (\ln y)^2 = \frac{x^2}{2} + C \;\Rightarrow$

$\ln y = \pm\sqrt{x^2/2 + C} \;\Rightarrow\; y = e^{\pm\sqrt{x^2/2+C}}$

9. $\dfrac{du}{dt} = 2 + 2u + t + tu \Rightarrow \dfrac{du}{dt} = (1+u)(2+t) \Rightarrow \displaystyle\int \frac{du}{1+u} = \int (2+t)dt \;\; [u \neq -1] \Rightarrow$

$\ln|1+u| = \frac{1}{2}t^2 + 2t + C \Rightarrow |1+u| = e^{t^2/2+2t+C} = Ke^{t^2/2+2t}$, where $K = e^C \Rightarrow$

$1 + u = \pm Ke^{t^2/2+2t} \Rightarrow u = -1 \pm Ke^{t^2/2+2t}$ where $K > 0$. $u = -1$ is also a solution, so

$u = -1 + Ae^{t^2/2+2t}$, where A is an arbitrary constant.

10. $\dfrac{dz}{dt} + e^{t+z} = 0 \Rightarrow \dfrac{dz}{dt} = -e^t e^z \Rightarrow \int e^{-z}\,dz = -\int e^t\,dt \Rightarrow -e^{-z} = -e^t + C \Rightarrow e^{-z} = e^t - C$

$\Rightarrow \dfrac{1}{e^z} = e^t - C \Rightarrow e^z = \dfrac{1}{e^t - C} \Rightarrow z = \ln\left(\dfrac{1}{e^t - C}\right) \Rightarrow z = -\ln(e^t - C)$

11. $\dfrac{dy}{dx} = y^2 + 1$, $y(1) = 0$. $\displaystyle\int \frac{dy}{y^2+1} = \int dx \Rightarrow \tan^{-1} y = x + C$. $y = 0$ when $x = 1$, so

$1 + C = \tan^{-1} 0 = 0 \Rightarrow C = -1$. Thus, $\tan^{-1} y = x - 1$ and $y = \tan(x-1)$.

12. $\dfrac{dy}{dx} = \dfrac{y\cos x}{1+y^2}$, $y(0) = 1$. $(1+y^2)\,dy = y\cos x\,dx \Rightarrow \dfrac{1+y^2}{y}dy = \cos x\,dx \Rightarrow$

$\int\left(\frac{1}{y} + y\right)dy = \int \cos x\,dx \Rightarrow \ln|y| + \frac{1}{2}y^2 = \sin x + C$. $y(0) = 1 \Rightarrow \ln 1 + \frac{1}{2} = \sin 0 + C \Rightarrow$

$C = \frac{1}{2}$, so $\ln|y| + \frac{1}{2}y^2 = \sin x + \frac{1}{2}$. We cannot solve explicitly for y.

13. $x\cos x = (2y + e^{3y})\,y' \Rightarrow x\cos x\,dx = (2y + e^{3y})\,dy \Rightarrow \int(2y + e^{3y})\,dy = \int x\cos x\,dx \Rightarrow$

$y^2 + \frac{1}{3}e^{3y} = x\sin x + \cos x + C$ [where the second integral is evaluated using integration by parts]. Now

$y(0) = 0 \Rightarrow 0 + \frac{1}{3} = 0 + 1 + C \Rightarrow C = -\frac{2}{3}$. Thus, a solution is $y^2 + \frac{1}{3}e^{3y} = x\sin x + \cos x - \frac{2}{3}$.

We cannot solve explicitly for y.

14. $\dfrac{dP}{dt} = \sqrt{Pt} \Rightarrow dP/\sqrt{P} = \sqrt{t}\,dt \Rightarrow \int P^{-1/2}\,dP = \int t^{1/2}\,dt \Rightarrow 2P^{1/2} = \frac{2}{3}t^{3/2} + C$.

$P(1) = 2 \Rightarrow 2\sqrt{2} = \frac{2}{3} + C \Rightarrow C = 2\sqrt{2} - \frac{2}{3}$, so $2P^{1/2} = \frac{2}{3}t^{3/2} + 2\sqrt{2} - \frac{2}{3} \Rightarrow$

$\sqrt{P} = \frac{1}{3}t^{3/2} + \sqrt{2} - \frac{1}{3} \Rightarrow P = \left(\frac{1}{3}t^{3/2} + \sqrt{2} - \frac{1}{3}\right)^2$.

15. $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}$, $u(0) = -5$. $\int 2u\,du = \int(2t + \sec^2 t)\,dt \Rightarrow u^2 = t^2 + \tan t + C$, where

$[u(0)]^2 = 0^2 + \tan 0 + C \Rightarrow C = (-5)^2 = 25$. Therefore, $u^2 = t^2 + \tan t + 25$, so $u = \pm\sqrt{t^2 + \tan t + 25}$.

Since $u(0) = -5$, we must have $u = -\sqrt{t^2 + \tan t + 25}$.

16. $\dfrac{dy}{dt} = te^y$, $y(1) = 0$. $\int e^{-y}\,dy = \int t\,dt \Rightarrow -e^{-y} = \frac{1}{2}t^2 + C$. Since $y(1) = 0$, $-e^0 = \frac{1}{2}\cdot 1^2 + C$. Therefore,

$C = -1 - \frac{1}{2} = -\frac{3}{2}$ and $-e^{-y} = \frac{1}{2}t^2 - \frac{3}{2}$. So $e^{-y} = \frac{3}{2} - \frac{1}{2}t^2 = \dfrac{3 - t^2}{2} \Rightarrow e^y = \dfrac{2}{3 - t^2} \Rightarrow$

$y = \ln 2 - \ln(3 - t^2)$ for $|t| < \sqrt{3}$.

17. $y' \tan x = a + y,\ 0 < x < \pi/2 \ \Rightarrow\ \dfrac{dy}{dx} = \dfrac{a+y}{\tan x} \ \Rightarrow\ \dfrac{dy}{a+y} = \cot x\, dx \quad [a + y \neq 0] \ \Rightarrow$

$\displaystyle\int \frac{dy}{a+y} = \int \frac{\cos x}{\sin x}\, dx \ \Rightarrow\ \ln|a+y| = \ln|\sin x| + C \ \Rightarrow$

$|a+y| = e^{\ln|\sin x|+C} = e^{\ln|\sin x|} \cdot e^C = e^C\,|\sin x| \ \Rightarrow\ a + y = K \sin x$, where $K = \pm e^C$. (In our derivation, K was nonzero, but we can restore the excluded case $y = -a$ by allowing K to be zero.) $y(\pi/3) = a \ \Rightarrow$

$a + a = K \sin\left(\dfrac{\pi}{3}\right) \ \Rightarrow\ 2a = K\,\dfrac{\sqrt{3}}{2} \ \Rightarrow\ K = \dfrac{4a}{\sqrt{3}}$. Thus, $a + y = \dfrac{4a}{\sqrt{3}} \sin x$ and so $y = \dfrac{4a}{\sqrt{3}} \sin x - a$.

18. $xy' + y = y^2 \ \Rightarrow\ x\,\dfrac{dy}{dx} = y^2 - y \ \Rightarrow\ x\,dy = (y^2 - y)\,dx \ \Rightarrow\ \dfrac{dy}{y^2 - y} = \dfrac{dx}{x} \ \Rightarrow$

$\displaystyle\int \frac{dy}{y(y-1)} = \int \frac{dx}{x} \quad [y \neq 0, 1] \ \Rightarrow\ \int \left(\frac{1}{y-1} - \frac{1}{y}\right) dy = \int \frac{dx}{x} \ \Rightarrow\ \ln|y-1| - \ln|y| = \ln|x| + C$

$\Rightarrow\ \ln\left|\dfrac{y-1}{y}\right| = \ln\left(e^C\,|x|\right) \ \Rightarrow\ \left|\dfrac{y-1}{y}\right| = e^C\,|x| \ \Rightarrow\ \dfrac{y-1}{y} = Kx$, where $K = \pm e^C \ \Rightarrow$

$1 - \dfrac{1}{y} = Kx \ \Rightarrow\ \dfrac{1}{y} = 1 - Kx \ \Rightarrow\ y = \dfrac{1}{1 - Kx}$. [The excluded cases, $y = 0$ and $y = 1$, are ruled out by the initial condition $y(1) = -1$.] Now $y(1) = -1 \ \Rightarrow\ -1 = \dfrac{1}{1-K} \ \Rightarrow\ 1 - K = -1 \ \Rightarrow\ K = 2$,

so $y = \dfrac{1}{1 - 2x}$.

19. $\dfrac{dy}{dx} = 4x^3 y,\ y(0) = 7.\ \dfrac{dy}{y} = 4x^3\,dx$ [if $y \neq 0$] $\Rightarrow\ \displaystyle\int \frac{dy}{y} = \int 4x^3\,dx \ \Rightarrow\ \ln|y| = x^4 + C \ \Rightarrow$

$e^{\ln|y|} = e^{x^4 + C} \ \Rightarrow\ |y| = e^{x^4} e^C \ \Rightarrow\ y = Ae^{x^4};\ y(0) = 7 \ \Rightarrow\ A = 7 \ \Rightarrow\ y = 7e^{x^4}$.

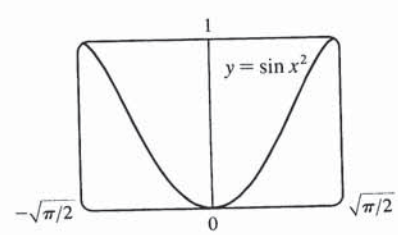
20. $\dfrac{dy}{dx} = \dfrac{y^2}{x^3},\ y(1) = 1.\ \displaystyle\int \frac{dy}{y^2} = \int \frac{dx}{x^3} \ \Rightarrow\ -\frac{1}{y} = -\frac{1}{2x^2} + C.\ y(1) = 1 \ \Rightarrow\ -1 = -\tfrac{1}{2} + C \ \Rightarrow$

$C = -\tfrac{1}{2}$. So $\dfrac{1}{y} = \dfrac{1}{2x^2} + \dfrac{1}{2} = \dfrac{2 + 2x^2}{2 \cdot 2x^2} \ \Rightarrow\ y = \dfrac{2x^2}{x^2 + 1}$.

21. (a) $y' = 2x\sqrt{1 - y^2} \ \Rightarrow\ \dfrac{dy}{dx} = 2x\sqrt{1 - y^2} \ \Rightarrow\ \dfrac{dy}{\sqrt{1 - y^2}} = 2x\,dx \ \Rightarrow\ \displaystyle\int \frac{dy}{\sqrt{1 - y^2}} = \int 2x\,dx \ \Rightarrow$

$\sin^{-1} y = x^2 + C$ for $-\frac{\pi}{2} \le x^2 + C \le \frac{\pi}{2}$.

(b) $y(0) = 0 \ \Rightarrow\ \sin^{-1} 0 = 0^2 + C \ \Rightarrow\ C = 0$, so $\sin^{-1} y = x^2$ and $y = \sin(x^2)$ for $-\sqrt{\pi/2} \le x \le \sqrt{\pi/2}$.



(c) For $\sqrt{1 - y^2}$ to be a real number, we must have $-1 \le y \le 1$; that is, $-1 \le y(0) \le 1$. Thus, the initial-value problem $y' = 2x\sqrt{1 - y^2},\ y(0) = 2$ does *not* have a solution.