5 THE INTEGRAL

5.1 Approximating and Computing Area

Preliminary Questions

1. What are the right and left endpoints if [2, 5] is divided into six subintervals?

SOLUTION If the interval [2, 5] is divided into six subintervals, the length of each subinterval is $\frac{5-2}{6} = \frac{1}{2}$. The right endpoints of the subintervals are then $\frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5$, while the left endpoints are $2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}$.

2. The interval [1, 5] is divided into eight subintervals.

(a) What is the left endpoint of the last subinterval?

(b) What are the right endpoints of the first two subintervals?

SOLUTION Note that each of the 8 subintervals has length $\frac{5-1}{8} = \frac{1}{2}$.

(a) The left endpoint of the last subinterval is $5 - \frac{1}{2} = \frac{9}{2}$.

(b) The right endpoints of the first two subintervals are $1 + \frac{1}{2} = \frac{3}{2}$ and $1 + 2\left(\frac{1}{2}\right) = 2$.

3. Which of the following pairs of sums are *not* equal?

(a) $\sum_{i=1}^{4} i, \quad \sum_{\ell=1}^{4} \ell$

(b) $\sum_{j=1}^{4} j^2, \quad \sum_{k=2}^{5} k^2$

(c) $\sum_{j=1}^{4} j, \quad \sum_{i=2}^{5} (i-1)$

(d) $\sum_{i=1}^{4} i(i+1), \quad \sum_{j=2}^{5} (j-1)j$

SOLUTION

(a) Only the name of the index variable has been changed, so these two sums *are* the same.

(b) These two sums are *not* the same; the second squares the numbers two through five while the first squares the numbers one through four.

(c) These two sums *are* the same. Note that when i ranges from two through five, the expression $i - 1$ ranges from one through four.

(d) These two sums *are* the same. Both sums are $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5$.

4. Explain: $\sum_{j=1}^{100} j = \sum_{j=0}^{100} j$ but $\sum_{j=1}^{100} 1$ is not equal to $\sum_{j=0}^{100} 1$.

SOLUTION The first term in the sum $\sum_{j=0}^{100} j$ is equal to zero, so it may be dropped. More specifically,

$$\sum_{j=0}^{100} j = 0 + \sum_{j=1}^{100} j = \sum_{j=1}^{100} j.$$

On the other hand, the first term in $\sum_{j=0}^{100} 1$ is not zero, so this term cannot be dropped. In particular,

$$\sum_{j=0}^{100} 1 = 1 + \sum_{j=1}^{100} 1 \neq \sum_{j=1}^{100} 1.$$

5. Explain why $L_{100} \geq R_{100}$ for $f(x) = x^{-2}$ on [3, 7].

SOLUTION On [3, 7], the function $f(x) = x^{-2}$ is a decreasing function; hence, for any subinterval of [3, 7], the function value at the left endpoint is larger than the function value at the right endpoint. Consequently, L_{100} must be larger than R_{100}.

Exercises

1. Figure 15 shows the velocity of an object over a 3-min interval. Determine the distance traveled over the intervals [0, 3] and [1, 2.5] (remember to convert from km/h to km/min).

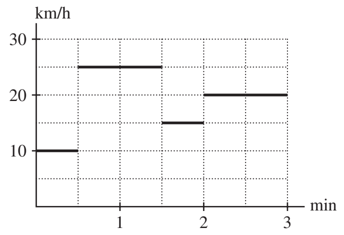


FIGURE 15

SOLUTION The distance traveled by the object can be determined by calculating the area underneath the velocity graph over the specified interval. During the interval [0, 3], the object travels

$$\left(\frac{10}{60}\right)\left(\frac{1}{2}\right)+\left(\frac{25}{60}\right)(1)+\left(\frac{15}{60}\right)\left(\frac{1}{2}\right)+\left(\frac{20}{60}\right)(1)=\frac{23}{24}\approx 0.96 \text{ km}.$$

During the interval [1, 2.5], it travels

$$\left(\frac{25}{60}\right)\left(\frac{1}{2}\right)+\left(\frac{15}{60}\right)\left(\frac{1}{2}\right)+\left(\frac{20}{60}\right)\left(\frac{1}{2}\right)=\frac{1}{2}=0.5 \text{ km}.$$

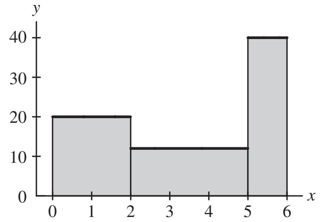
2. An ostrich (Figure 16) runs with velocity 20 km/h for 2 minutes, 12 km/h for 3 minutes, and 40 km/h for another minute. Compute the total distance traveled and indicate with a graph how this quantity can be interpreted as an area.

FIGURE 16 Ostriches can reach speeds as high as 70 km/h.

SOLUTION The total distance traveled by the ostrich is

$$\left(\frac{20}{60}\right)(2)+\left(\frac{12}{60}\right)(3)+\left(\frac{40}{60}\right)(1)=\frac{2}{3}+\frac{3}{5}+\frac{2}{3}=\frac{29}{15}$$

km. This distance is the area under the graph below which shows the ostrich's velocity as a function of time.



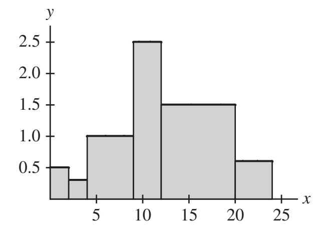
3. A rainstorm hit Portland, Maine, in October 1996, resulting in record rainfall. The rainfall rate $R(t)$ on October 21 is recorded, in centimeters per hour, in the following table, where t is the number of hours since midnight. Compute the total rainfall during this 24-hour period and indicate on a graph how this quantity can be interpreted as an area.

t (h)	0–2	2–4	4–9	9–12	12–20	20–24
$R(t)$ (cm)	0.5	0.3	1.0	2.5	1.5	0.6

SOLUTION Over each interval, the total rainfall is the time interval in hours times the rainfall in centimeters per hour. Thus

$$R=2(0.5)+2(0.3)+5(1.0)+3(2.5)+8(1.5)+4(0.6)=28.5 \text{ cm}.$$

The figure below is a graph of the rainfall as a function of time. The area of the shaded region represents the total rainfall.



4. The velocity of an object is $v(t) = 12t$ m/s. Use Eq. (2) and geometry to find the distance traveled over the time intervals [0, 2] and [2, 5].

SOLUTION By equation Eq. (2), the distance traveled over the time interval $[a, b]$ is

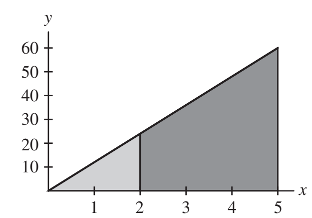
$$\int_a^b v(t)\,dt = \int_a^b 12t\,dt;$$

that is, the distance traveled is the area under the graph of the velocity function over the interval $[a, b]$. The graph below shows the area under the velocity function $v(t) = 12t$ m/s over the intervals [0, 2] and [2, 5]. Over the interval [0, 2], the area is a triangle of base 2 and height 24; therefore, the distance traveled is

$$\frac{1}{2}(2)(24) = 24 \text{ meters.}$$

Over the interval [2, 5], the area is a trapezoid of height 3 and base lengths 24 and 60; therefore, the distance traveled is

$$\frac{1}{2}(3)(24 + 60) = 126 \text{ meters.}$$



5. Compute R_5 and L_5 over [0, 1] using the following values.

x	0	0.2	0.4	0.6	0.8	1
$f(x)$	50	48	46	44	42	40

SOLUTION $\Delta x = \frac{1-0}{5} = 0.2$. Thus,

$$L_5 = 0.2\,(50 + 48 + 46 + 44 + 42) = 0.2(230) = 46,$$

and

$$R_5 = 0.2\,(48 + 46 + 44 + 42 + 40) = 0.2(220) = 44.$$

The average is

$$\frac{46 + 44}{2} = 45.$$

This estimate is frequently referred to as the *Trapezoidal Approximation*.

6. Compute R_6, L_6, and M_3 to estimate the distance traveled over [0, 3] if the velocity at half-second intervals is as follows:

t (s)	0	0.5	1	1.5	2	2.5	3
v (m/s)	0	12	18	25	20	14	20

SOLUTION For R_6 and L_6, $\Delta t = \frac{3-0}{6} = 0.5$. For M_3, $\Delta t = \frac{3-0}{3} = 1$. Then

$$R_6 = 0.5 \text{ s}\,(12 + 18 + 25 + 20 + 14 + 20) \text{ m/s} = 0.5(109) \text{ m} = 54.5 \text{ m},$$

$$L_6 = 0.5\,\text{sec}\,(0 + 12 + 18 + 25 + 20 + 14)\ \text{m/sec} = 0.5(89)\ \text{m} = 44.5\ \text{m},$$

and

$$M_3 = 1\,\text{sec}\,(12 + 25 + 14)\ \text{m/sec} = 51\ \text{m}.$$

7. Let $f(x) = 2x + 3$.

(a) Compute R_6 and L_6 over $[0, 3]$.

(b) Use geometry to find the exact area A and compute the errors $|A - R_6|$ and $|A - L_6|$ in the approximations.

SOLUTION Let $f(x) = 2x + 3$ on $[0, 3]$.

(a) We partition $[0, 3]$ into 6 equally-spaced subintervals. The left endpoints of the subintervals are $\left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}\right\}$ whereas the right endpoints are $\left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\right\}$.

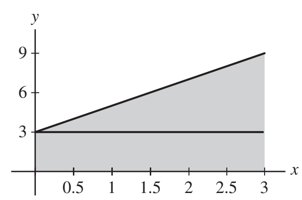
- Let $a = 0$, $b = 3$, $n = 6$, $\Delta x = (b - a)/n = \frac{1}{2}$, and $x_k = a + k\Delta x$, $k = 0, 1, \ldots, 5$ (left endpoints). Then

$$L_6 = \sum_{k=0}^{5} f(x_k)\Delta x = \Delta x \sum_{k=0}^{5} f(x_k) = \frac{1}{2}(3 + 4 + 5 + 6 + 7 + 8) = 16.5.$$

- With $x_k = a + k\Delta x$, $k = 1, 2, \ldots, 6$ (right endpoints), we have

$$R_6 = \sum_{k=1}^{6} f(x_k)\Delta x = \Delta x \sum_{k=1}^{6} f(x_k) = \frac{1}{2}(4 + 5 + 6 + 7 + 8 + 9) = 19.5.$$

(b) Via geometry (see figure below), the exact area is $A = \frac{1}{2}(3)(6) + 3^2 = 18$. Thus, L_6 underestimates the true area ($L_6 - A = -1.5$), while R_6 overestimates the true area ($R_6 - A = +1.5$).



8. Repeat Exercise 7 for $f(x) = 20 - 3x$ over $[2, 4]$.

SOLUTION Let $f(x) = 20 - 3x$ on $[2, 4]$.

(a) We partition $[2, 4]$ into 6 equally-spaced subintervals. The left endpoints of the subintervals are $\left\{2, \frac{7}{3}, \frac{8}{3}, 3, \frac{10}{3}, \frac{11}{3}\right\}$ whereas the right endpoints are $\left\{\frac{7}{3}, \frac{8}{3}, 3, \frac{10}{3}, \frac{11}{3}, 3\right\}$.

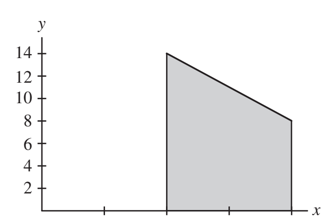
- Let $a = 2$, $b = 4$, $n = 6$, $\Delta x = (b - a)/n = \frac{1}{3}$, and $x_k = a + k\Delta x$, $k = 0, 1, \ldots, 5$ (left endpoints). Then

$$L_6 = \sum_{k=0}^{5} f(x_k)\Delta x = \Delta x \sum_{k=0}^{5} f(x_k) = \frac{1}{3}(14 + 13 + 12 + 11 + 10 + 9) = 23.$$

- With $x_k = a + k\Delta x$, $k = 1, 2, \ldots, 6$ (right endpoints), we have

$$R_6 = \sum_{k=1}^{6} f(x_k)\Delta x = \Delta x \sum_{k=1}^{6} f(x_k) = \frac{1}{3}(13 + 12 + 11 + 10 + 9 + 8) = 21.$$

(b) Via geometry (see figure below), the exact area is $A = \frac{1}{2}(2)(14 + 8) = 22$. Thus, L_6 overestimates the true area ($L_6 - A = 1$), while R_6 underestimates the true area ($R_6 - A = -1$).



9. Calculate R_3 and L_3

$$\text{for } f(x) = x^2 - x + 4 \quad \text{over } [1, 4]$$

Then sketch the graph of f and the rectangles that make up each approximation. Is the area under the graph larger or smaller than R_3? Is it larger or smaller than L_3?

SOLUTION Let $f(x) = x^2 - x + 4$ and set $a = 1$, $b = 4$, $n = 3$, $\Delta x = (b - a)/n = (4 - 1)/3 = 1$.

(a) Let $x_k = a + k\Delta x$, $k = 0, 1, 2, 3$.

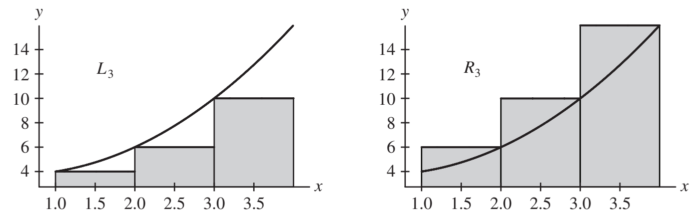
- Selecting the left endpoints of the subintervals, x_k, $k = 0, 1, 2$, or $\{1, 2, 3\}$, we have

$$L_3 = \sum_{k=0}^{2} f(x_k)\Delta x = \Delta x \sum_{k=0}^{2} f(x_k) = (1)(4 + 6 + 10) = 20.$$

- Selecting the right endpoints of the subintervals, x_k, $k = 1, 2, 3$, or $\{2, 3, 4\}$, we have

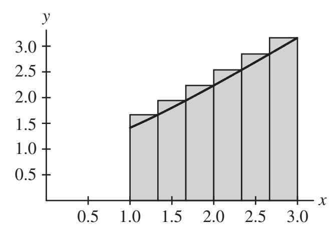
$$R_3 = \sum_{k=1}^{3} f(x_k)\Delta x = \Delta x \sum_{k=1}^{3} f(x_k) = (1)(6 + 10 + 16) = 32.$$

(b) Here are figures of the three rectangles that approximate the area under the curve $f(x)$ over the interval $[1, 4]$. Clearly, the area under the graph is larger than L_3 but smaller than R_3.



10. Let $f(x) = \sqrt{x^2 + 1}$ and $\Delta x = \frac{1}{3}$. Sketch the graph of $f(x)$ and draw the right-endpoint rectangles whose area is represented by the sum $\sum_{i=1}^{6} f(1 + i\Delta x)\Delta x$.

SOLUTION Because $\Delta x = \frac{1}{3}$ and the sum evaluates f at $1 + i\Delta x$ for i from 1 through 6, it follows that the interval over which we are considering f is $[1, 3]$. The sketch of f together with the six rectangles is shown below.



11. Estimate R_3, M_3, and L_6 over $[0, 1.5]$ for the function in Figure 17.

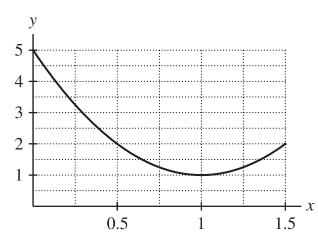


FIGURE 17

SOLUTION Let $f(x)$ on $[0, \frac{3}{2}]$ be given by Figure 17. For $n = 3$, $\Delta x = (\frac{3}{2} - 0)/3 = \frac{1}{2}$, $\{x_k\}_{k=0}^{3} = \left\{0, \frac{1}{2}, 1, \frac{3}{2}\right\}$. Therefore

$$R_3 = \frac{1}{2} \sum_{k=1}^{3} f(x_k) = \frac{1}{2}(2 + 1 + 2) = 2.5,$$

$$M_3 = \frac{1}{2}\sum_{k=1}^{6} f\left(x_k - \frac{1}{2}\Delta x\right) = \frac{1}{2}(3.25 + 1.25 + 1.25) = 2.875.$$

For $n = 6$, $\Delta x = (\frac{3}{2} - 0)/6 = \frac{1}{4}$, $\{x_k\}_{k=0}^{6} = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}\right\}$. Therefore

$$L_6 = \frac{1}{4}\sum_{k=0}^{5} f(x_k) = \frac{1}{4}(5 + 3.25 + 2 + 1.25 + 1 + 1.25) = 3.4375.$$

12. Calculate the area of the shaded rectangles in Figure 18. Which approximation do these rectangles represent?

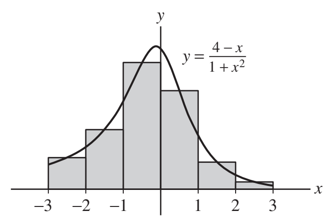


FIGURE 18

SOLUTION Each rectangle in Figure 18 has a width of 1 and the height is taken as the value of the function at the midpoint of the interval. Thus, the area of the shaded rectangles is

$$1\left(\frac{26}{29} + \frac{22}{13} + \frac{18}{5} + \frac{14}{5} + \frac{10}{13} + \frac{6}{29}\right) = \frac{18784}{1885} \approx 9.965.$$

Because there are six rectangles and the height of each rectangle is taken as the value of the function at the midpoint of the interval, the shaded rectangles represent the approximation M_6 to the area under the curve.

In Exercises 13–20, calculate the approximation for the given function and interval.

13. R_3, $f(x) = 7 - x$, $[3, 5]$

SOLUTION Let $f(x) = 7 - x$ on $[3, 5]$. For $n = 3$, $\Delta x = (5 - 3)/3 = \frac{2}{3}$, and $\{x_k\}_{k=0}^{3} = \left\{3, \frac{11}{3}, \frac{13}{3}, 5\right\}$. Therefore

$$R_3 = \frac{2}{3}\sum_{k=1}^{3}(7 - x_k)$$
$$= \frac{2}{3}\left(\frac{10}{3} + \frac{8}{3} + 2\right) = \frac{2}{3}(8) = \frac{16}{3}.$$

14. L_6, $f(x) = \sqrt{6x + 2}$, $[1, 3]$

SOLUTION Let $f(x) = \sqrt{6x + 2}$ on $[1, 3]$. For $n = 6$, $\Delta x = (3 - 1)/6 = \frac{1}{3}$, and $\{x_k\}_{k=0}^{6} = \left\{1, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, 3\right\}$. Therefore

$$L_6 = \frac{1}{3}\sum_{k=0}^{5}\sqrt{6x_k + 2}$$
$$= \frac{1}{3}\left(\sqrt{8} + \sqrt{10} + \sqrt{12} + \sqrt{14} + 4 + \sqrt{18}\right) \approx 7.146368.$$

15. M_6, $f(x) = 4x + 3$, $[5, 8]$

SOLUTION Let $f(x) = 4x + 3$ on $[5, 8]$. For $n = 6$, $\Delta x = (8 - 5)/6 = \frac{1}{2}$, and $\{x_k^*\}_{k=0}^{5} = \{5.25, 5.75, 6.25, 6.75, 7.25, 7.75\}$. Therefore,

$$M_6 = \frac{1}{2}\sum_{k=0}^{5}\left(4x_k^* + 3\right)$$
$$= \frac{1}{2}(24 + 26 + 28 + 30 + 32 + 34)$$
$$= \frac{1}{2}(174) = 87.$$

16. R_5, $f(x) = x^2 + x$, $[-1, 1]$

SOLUTION Let $f(x) = x^2 + x$ on $[-1, 1]$. For $n = 5$, $\Delta x = (1 - (-1))/5 = \frac{2}{5}$, and $\{x_k\}_{k=0}^{5} = \left\{-1, -\frac{3}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, 1\right\}$. Therefore

$$R_5 = \frac{2}{5}\sum_{k=1}^{5}(x_k^2 + x_k) = \frac{2}{5}\left(\left(\frac{9}{25} - \frac{3}{5}\right) + \left(\frac{1}{25} - \frac{1}{5}\right) + \left(\frac{1}{25} + \frac{1}{5}\right) + \left(\frac{9}{25} + \frac{3}{5}\right) + 2\right)$$
$$= \frac{2}{5}\left(\frac{14}{5}\right) = \frac{28}{25}.$$

17. L_6, $f(x) = x^2 + 3|x|$, $[-2, 1]$

SOLUTION Let $f(x) = x^2 + 3\,|x|$ on $[-2, 1]$. For $n = 6$, $\Delta x = (1 - (-2))/6 = \frac{1}{2}$, and $\{x_k\}_{k=0}^{6} = \{-2, -1.5, -1, -0.5, 0, 0.5, 1\}$. Therefore

$$L_6 = \frac{1}{2}\sum_{k=0}^{5}(x_k^2 + 3\,|x_k|) = \frac{1}{2}(10 + 6.75 + 4 + 1.75 + 0 + 1.75) = 12.125.$$

18. M_4, $f(x) = \sqrt{x}$, $[3, 5]$

SOLUTION Let $f(x) = \sqrt{x}$ on $[3, 5]$. For $n = 4$, $\Delta x = (5 - 3)/4 = \frac{1}{2}$, and $\{x_k^*\}_{k=0}^{3} = \{\frac{13}{4}, \frac{15}{4}, \frac{17}{4}, \frac{19}{4}\}$. Therefore,

$$M_4 = \frac{1}{2}\sum_{k=0}^{3}\sqrt{x_k^*}$$
$$= \frac{1}{2}\left(\frac{\sqrt{13}}{2} + \frac{\sqrt{15}}{2} + \frac{\sqrt{17}}{2} + \frac{\sqrt{19}}{2}\right) \approx 3.990135.$$

19. L_4, $f(x) = \cos^2 x$, $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

SOLUTION Let $f(x) = \cos^2 x$ on $[\frac{\pi}{6}, \frac{\pi}{2}]$. For $n = 4$,

$$\Delta x = \frac{(\pi/2 - \pi/6)}{4} = \frac{\pi}{12} \qquad \text{and} \qquad \{x_k\}_{k=0}^{4} = \left\{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{\pi}{2}\right\}.$$

Therefore

$$L_4 = \frac{\pi}{12}\sum_{k=0}^{3}\cos^2 x_k \approx 0.410236.$$

20. M_5, $f(x) = \ln x$, $[1, 3]$

SOLUTION Let $f(x) = \ln x$ on $[1, 3]$. For $n = 5$, $\Delta x = (3 - 1)/5 = \frac{2}{5}$, and $\{x_k^*\}_{k=0}^{4} = \{\frac{6}{5}, \frac{8}{5}, 2, \frac{12}{5}, \frac{14}{5}\}$. Therefore,

$$M_5 = \frac{2}{5}\sum_{k=0}^{4}\ln x_k^*$$
$$= \frac{2}{5}\left(\ln\frac{6}{5} + \ln\frac{8}{5} + \ln 2 + \ln\frac{12}{5} + \ln\frac{14}{5}\right) \approx 1.300224.$$

In Exercises 21–26, write the sum in summation notation.

21. $4^7 + 5^7 + 6^7 + 7^7 + 8^7$

SOLUTION The first term is 4^7, and the last term is 8^7, so it seems the kth term is k^7. Therefore, the sum is:

$$\sum_{k=4}^{8} k^7.$$

22. $(2^2 + 2) + (3^2 + 3) + (4^2 + 4) + (5^2 + 5)$

SOLUTION The first term is $2^2 + 2$, and the last term is $5^2 + 5$, so it seems that the sum limits are 2 and 5, and the kth term is $k^2 + k$. Therefore, the sum is:

$$\sum_{k=2}^{5}(k^2 + k).$$

23. $(2^2+2)+(2^3+2)+(2^4+2)+(2^5+2)$

SOLUTION The first term is 2^2+2, and the last term is 2^5+2, so it seems the sum limits are 2 and 5, and the kth term is 2^k+2. Therefore, the sum is:

$$\sum_{k=2}^{5}(2^k+2).$$

24. $\sqrt{1+1^3}+\sqrt{2+2^3}+\cdots+\sqrt{n+n^3}$

SOLUTION The first term is $\sqrt{1+1^3}$ and the last term is $\sqrt{n+n^3}$, so it seems the summation limits are 1 through n, and the k-th term is $\sqrt{k+k^3}$. Therefore, the sum is

$$\sum_{k=1}^{n}\sqrt{k+k^3}.$$

25. $\frac{1}{2\cdot 3}+\frac{2}{3\cdot 4}+\cdots+\frac{n}{(n+1)(n+2)}$

SOLUTION The first summand is $\frac{1}{(1+1)\cdot(1+2)}$. This shows us

$$\frac{1}{2\cdot 3}+\frac{2}{3\cdot 4}+\cdots+\frac{n}{(n+1)(n+2)}=\sum_{i=1}^{n}\frac{i}{(i+1)(i+2)}.$$

26. $e^{\pi}+e^{\pi/2}+e^{\pi/3}+\cdots+e^{\pi/n}$

SOLUTION The first term is $e^{\pi/1}$ and the last term is $e^{\pi/n}$, so it seems the sum limits are 1 and n and the kth term is $e^{\pi/k}$. Therefore, the sum is

$$\sum_{k=1}^{n}e^{\pi/k}.$$

27. Calculate the sums:

(a) $\sum_{i=1}^{5}9$ **(b)** $\sum_{i=0}^{5}4$ **(c)** $\sum_{k=2}^{4}k^3$

SOLUTION

(a) $\sum_{i=1}^{5}9=9+9+9+9+9=45$. Alternatively, $\sum_{i=1}^{5}9=9\sum_{i=1}^{5}1=(9)(5)=45$.

(b) $\sum_{i=0}^{5}4=4+4+4+4+4+4=24$. Alternatively, $\sum_{i=0}^{5}4=4\sum_{i=0}^{5}=(4)(6)=24$.

(c) $\sum_{k=2}^{4}k^3=2^3+3^3+4^3=99$. Alternatively,

$$\sum_{k=2}^{4}k^3=\left(\sum_{k=1}^{4}k^3\right)-\left(\sum_{k=1}^{1}k^3\right)=\left(\frac{4^4}{4}+\frac{4^3}{2}+\frac{4^2}{4}\right)-\left(\frac{1^4}{4}+\frac{1^3}{2}+\frac{1^2}{4}\right)=99.$$

28. Calculate the sums:

(a) $\sum_{j=3}^{4}\sin\left(j\frac{\pi}{2}\right)$ **(b)** $\sum_{k=3}^{5}\frac{1}{k-1}$ **(c)** $\sum_{j=0}^{2}3^{j-1}$

SOLUTION

(a) $\sum_{j=3}^{4}\sin\left(\frac{j\pi}{2}\right)=\sin\left(\frac{3\pi}{2}\right)+\sin\left(\frac{4\pi}{2}\right)=-1+0=-1.$

(b) $\sum_{k=3}^{5}\frac{1}{k-1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{13}{12}.$

(c) $\sum_{j=0}^{2}3^{j-1}=\frac{1}{3}+1+3=\frac{13}{3}.$

29. Let $b_1 = 4$, $b_2 = 1$, $b_3 = 2$, and $b_4 = -4$. Calculate:

(a) $\sum_{i=2}^{4} b_i$ **(b)** $\sum_{j=1}^{2} (2^{b_j} - b_j)$ **(c)** $\sum_{k=1}^{3} kb_k$

SOLUTION

(a) $\sum_{i=2}^{4} b_i = b_2 + b_3 + b_4 = 1 + 2 + (-4) = -1.$

(b) $\sum_{j=1}^{2} \left(2^{b_j} - b_j\right) = (2^4 - 4) + (2^1 - 1) = 13.$

(c) $\sum_{k=1}^{3} kb_k = 1(4) + 2(1) + 3(2) = 12.$

30. Assume that $a_1 = -5$, $\sum_{i=1}^{10} a_i = 20$, and $\sum_{i=1}^{10} b_i = 7$. Calculate:

(a) $\sum_{i=1}^{10} (4a_i + 3)$ **(b)** $\sum_{i=2}^{10} a_i$ **(c)** $\sum_{i=1}^{10} (2a_i - 3b_i)$

SOLUTION

(a) $\sum_{i=1}^{10} (4a_i + 3) = 4\sum_{i=1}^{10} a_i + 3\sum_{i=1}^{10} 1 = 4(20) + 3(10) = 110.$

(b) $\sum_{i=2}^{10} a_i = \sum_{i=1}^{10} a_i - a_1 = 20 - (-5) = 25.$

(c) $\sum_{i=1}^{10} (2a_i - 3b_i) = 2\sum_{i=1}^{10} a_i - 3\sum_{i=1}^{10} b_i = 2(20) - 3(7) = 19.$

31. Calculate $\sum_{j=101}^{200} j$. *Hint:* Write as a difference of two sums and use formula (3).

SOLUTION

$$\sum_{j=101}^{200} j = \sum_{j=1}^{200} j - \sum_{j=1}^{100} j = \left(\frac{200^2}{2} + \frac{200}{2}\right) - \left(\frac{100^2}{2} + \frac{100}{2}\right) = 20100 - 5050 = 15050.$$

32. Calculate $\sum_{j=1}^{30} (2j + 1)^2$. *Hint:* Expand and use formulas (3)–(4).

SOLUTION

$$\sum_{j=1}^{30} (2j + 1)^2 = 4\sum_{j=1}^{30} j^2 + 4\sum_{j=1}^{30} j + \sum_{j=1}^{30} 1$$
$$= 4\left(\frac{30^3}{3} + \frac{30^2}{2} + \frac{30}{6}\right) + 4\left(\frac{30^2}{2} + \frac{30}{2}\right) + 30$$
$$= 39{,}710.$$

In Exercises 33–40, use linearity and formulas (3)–(5) to rewrite and evaluate the sums.

33. $\sum_{j=1}^{20} 8j^3$

SOLUTION $\sum_{j=1}^{20} 8j^3 = 8\sum_{j=1}^{20} j^3 = 8\left(\frac{20^4}{4} + \frac{20^3}{2} + \frac{20^2}{4}\right) = 8(44{,}100) = 352{,}800.$

34. $\sum_{k=1}^{30}(4k-3)$

SOLUTION

$$\sum_{k=1}^{30}(4k-3) = 4\sum_{k=1}^{30}k - 3\sum_{k=1}^{30}1$$
$$= 4\left(\frac{30^2}{2}+\frac{30}{2}\right) - 3(30) = 4(465) - 90 = 1770.$$

35. $\sum_{n=51}^{150} n^2$

SOLUTION

$$\sum_{n=51}^{150} n^2 = \sum_{n=1}^{150} n^2 - \sum_{n=1}^{50} n^2$$
$$= \left(\frac{150^3}{3}+\frac{150^2}{2}+\frac{150}{6}\right) - \left(\frac{50^3}{3}+\frac{50^2}{2}+\frac{50}{6}\right)$$
$$= 1{,}136{,}275 - 42{,}925 = 1{,}093{,}350.$$

36. $\sum_{k=101}^{200} k^3$

SOLUTION

$$\sum_{k=101}^{200} k^3 = \sum_{k=1}^{200} k^3 - \sum_{k=1}^{100} k^3$$
$$= \left(\frac{200^4}{4}+\frac{200^3}{2}+\frac{200^2}{4}\right) - \left(\frac{100^4}{4}+\frac{100^3}{2}+\frac{100^2}{4}\right)$$
$$= 404{,}010{,}000 - 25{,}502{,}500 = 378{,}507{,}500.$$

37. $\sum_{j=0}^{50} j(j-1)$

SOLUTION

$$\sum_{j=0}^{50} j(j-1) = \sum_{j=0}^{50}(j^2-j) = \sum_{j=0}^{50} j^2 - \sum_{j=0}^{50} j$$
$$= \left(\frac{50^3}{3}+\frac{50^2}{2}+\frac{50}{6}\right) - \left(\frac{50^2}{2}+\frac{50}{2}\right) = \frac{50^3}{3} - \frac{50}{3} = \frac{124{,}950}{3} = 41{,}650.$$

The power sum formula is usable because $\sum_{j=0}^{50} j(j-1) = \sum_{j=1}^{50} j(j-1)$.

38. $\sum_{j=2}^{30}\left(6j+\frac{4j^2}{3}\right)$

SOLUTION

$$\sum_{j=2}^{30}\left(6j+\frac{4j^2}{3}\right) = 6\sum_{j=2}^{30} j + \frac{4}{3}\sum_{j=2}^{30} j^2 = 6\left(\sum_{j=1}^{30} j - \sum_{j=1}^{1} j\right) + \frac{4}{3}\left(\sum_{j=1}^{30} j^2 - \sum_{j=1}^{1} j^2\right)$$
$$= 6\left(\frac{30^2}{2}+\frac{30}{2}-1\right) + \frac{4}{3}\left(\frac{30^3}{3}+\frac{30^2}{2}+\frac{30}{6}-1\right)$$
$$= 6\,(464) + \frac{4}{3}\,(9454) = 2784 + \frac{37{,}816}{3} = \frac{46{,}168}{3}.$$

39. $\sum_{m=1}^{30}(4-m)^3$

SOLUTION

$$\begin{aligned}\sum_{m=1}^{30}(4-m)^3 &= \sum_{m=1}^{30}(64-48m+12m^2-m^3)\\ &= 64\sum_{m=1}^{30}1-48\sum_{m=1}^{30}m+12\sum_{m=1}^{30}m^2-\sum_{m=1}^{30}m^3\\ &= 64(30)-48\frac{(30)(31)}{2}+12\left(\frac{30^3}{3}+\frac{30^2}{2}+\frac{30}{6}\right)-\left(\frac{30^4}{4}+\frac{30^3}{2}+\frac{30^2}{4}\right)\\ &= 1920-22{,}320+113{,}460-216{,}225=-123{,}165.\end{aligned}$$

40. $\sum_{m=1}^{20}\left(5+\frac{3m}{2}\right)^2$

SOLUTION

$$\begin{aligned}\sum_{m=1}^{20}\left(5+\frac{3m}{2}\right)^2 &= 25\sum_{m=1}^{20}1+15\sum_{m=1}^{20}m+\frac{9}{4}\sum_{m=1}^{20}m^2\\ &= 25(20)+15\left(\frac{20^2}{2}+\frac{20}{2}\right)+\frac{9}{4}\left(\frac{20^3}{3}+\frac{20^2}{2}+\frac{20}{6}\right)\\ &= 500+15(210)+\frac{9}{4}(2870)=10107.5.\end{aligned}$$

In Exercises 41–44, use formulas (3)–(5) to evaluate the limit.

41. $\lim_{N\to\infty}\sum_{i=1}^{N}\frac{i}{N^2}$

SOLUTION Let $s_N=\sum_{i=1}^{N}\frac{i}{N^2}$. Then,

$$s_N=\sum_{i=1}^{N}\frac{i}{N^2}=\frac{1}{N^2}\sum_{i=1}^{N}i=\frac{1}{N^2}\left(\frac{N^2}{2}+\frac{N}{2}\right)=\frac{1}{2}+\frac{1}{2N}.$$

Therefore, $\lim_{N\to\infty}s_N=\frac{1}{2}$.

42. $\lim_{N\to\infty}\sum_{j=1}^{N}\frac{j^3}{N^4}$

SOLUTION Let $s_N=\sum_{j=1}^{N}\frac{j^3}{N^4}$. Then

$$s_N=\frac{1}{N^4}\sum_{j=1}^{N}j^3=\frac{1}{N^4}\left(\frac{N^4}{4}+\frac{N^3}{2}+\frac{N^2}{4}\right)=\frac{1}{4}+\frac{1}{2N}+\frac{1}{4N^2}.$$

Therefore, $\lim_{N\to\infty}s_N=\frac{1}{4}$.

43. $\lim_{N\to\infty} \sum_{i=1}^{N} \frac{i^2 - i + 1}{N^3}$

SOLUTION Let $s_N = \sum_{i=1}^{N} \frac{i^2 - i + 1}{N^3}$. Then

$$s_N = \sum_{i=1}^{N} \frac{i^2 - i + 1}{N^3} = \frac{1}{N^3}\left[\left(\sum_{i=1}^{N} i^2\right) - \left(\sum_{i=1}^{N} i\right) + \left(\sum_{i=1}^{N} 1\right)\right]$$
$$= \frac{1}{N^3}\left[\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) - \left(\frac{N^2}{2} + \frac{N}{2}\right) + N\right] = \frac{1}{3} + \frac{2}{3N^2}.$$

Therefore, $\lim_{N\to\infty} s_N = \frac{1}{3}$.

44. $\lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{i^3}{N^4} - \frac{20}{N}\right)$

SOLUTION Let $s_N = \sum_{i=1}^{N} \left(\frac{i^3}{N^4} - \frac{20}{N}\right)$. Then

$$s_N = \frac{1}{N^4}\sum_{i=1}^{N} i^3 - \frac{20}{N}\sum_{i=1}^{N} 1 = \frac{1}{N^4}\left(\frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{4}\right) - 20 = \frac{1}{4} + \frac{1}{2N} + \frac{1}{4N^2} - 20.$$

Therefore, $\lim_{N\to\infty} s_N = \frac{1}{4} - 20 = -\frac{79}{4}$.

In Exercises 45–50, calculate the limit for the given function and interval. Verify your answer by using geometry.

45. $\lim_{N\to\infty} R_N, \quad f(x) = 9x, \quad [0, 2]$

SOLUTION Let $f(x) = 9x$ on $[0, 2]$. Let N be a positive integer and set $a = 0$, $b = 2$, and $\Delta x = (b - a)/N = (2 - 0)/N = 2/N$. Also, let $x_k = a + k\Delta x = 2k/N$, $k = 1, 2, \ldots, N$ be the right endpoints of the N subintervals of $[0, 2]$. Then

$$R_N = \Delta x \sum_{k=1}^{N} f(x_k) = \frac{2}{N}\sum_{k=1}^{N} 9\left(\frac{2k}{N}\right) = \frac{36}{N^2}\sum_{k=1}^{N} k = \frac{36}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) = 18 + \frac{18}{N}.$$

The area under the graph is

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(18 + \frac{18}{N}\right) = 18.$$

The region under the graph is a triangle with base 2 and height 18. The area of the region is then $\frac{1}{2}(2)(18) = 18$, which agrees with the value obtained from the limit of the right-endpoint approximations.

46. $\lim_{N\to\infty} R_N, \quad f(x) = 3x + 6, \quad [1, 4]$

SOLUTION Let $f(x) = 3x + 6$ on $[1, 4]$. Let N be a positive integer and set $a = 1$, $b = 4$, and $\Delta x = (b - a)/N = (4 - 1)/N = 3/N$. Also, let $x_k = a + k\Delta x = 1 + 3k/N$, $k = 1, 2, \ldots, N$ be the right endpoints of the N subintervals of $[1, 4]$. Then

$$R_N = \Delta x \sum_{k=1}^{N} f(x_k) = \frac{3}{N}\sum_{k=1}^{N}\left(9 + \frac{9k}{N}\right)$$
$$= \frac{27}{N}\sum_{k=1}^{N} 1 + \frac{27}{N^2}\sum_{k=1}^{N} j = \frac{27}{N}(N) + \frac{27}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right)$$
$$= \frac{81}{2} + \frac{27}{2N}.$$

The area under the graph is

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{81}{2} + \frac{27}{2N}\right) = \frac{81}{2}.$$

The region under the graph is a trapezoid with base width 3 and heights 9 and 18. The area of the region is then $\frac{1}{2}(3)(9+18) = \frac{81}{2}$, which agrees with the value obtained from the limit of the right-endpoint approximations.

47. $\lim_{N\to\infty} L_N, \quad f(x) = \frac{1}{2}x + 2, \quad [0, 4]$

SOLUTION Let $f(x) = \frac{1}{2}x + 2$ on $[0, 4]$. Let $N > 0$ be an integer, and set $a = 0$, $b = 4$, and $\Delta x = (4-0)/N = \frac{4}{N}$. Also, let $x_k = 0 + k\Delta x = \frac{4k}{N}$, $k = 0, 1, \dots, N-1$ be the left endpoints of the N subintervals. Then

$$L_N = \Delta x \sum_{k=0}^{N-1} f(x_k) = \frac{4}{N} \sum_{k=0}^{N-1} \left(\frac{1}{2}\left(\frac{4k}{N}\right) + 2\right) = \frac{8}{N} \sum_{k=0}^{N-1} 1 + \frac{8}{N^2} \sum_{k=0}^{N-1} k$$
$$= 8 + \frac{8}{N^2}\left(\frac{(N-1)^2}{2} + \frac{N-1}{2}\right) = 12 - \frac{4}{N}.$$

The area under the graph is

$$\lim_{N\to\infty} L_N = 12.$$

The region under the curve over $[0, 4]$ is a trapezoid with base width 4 and heights 2 and 4. From this, we get that the area is $\frac{1}{2}(4)(2+4) = 12$, which agrees with the answer obtained from the limit of the left-endpoint approximations.

48. $\lim_{N\to\infty} L_N, \quad f(x) = 4x - 2, \quad [1, 3]$

SOLUTION Let $f(x) = 4x - 2$ on $[1, 3]$. Let $N > 0$ be an integer, and set $a = 1$, $b = 3$, and $\Delta x = (3-1)/N = \frac{2}{N}$. Also, let $x_k = a + k\Delta x = 1 + \frac{2k}{N}$, $k = 0, 1, \dots, N-1$ be the left endpoints of the N subintervals. Then

$$L_N = \Delta x \sum_{k=0}^{N-1} f(x_k) = \frac{2}{N} \sum_{k=0}^{N-1} \left(\frac{8k}{N} + 2\right) = \frac{16}{N^2} \sum_{k=0}^{N-1} k + \frac{4}{N} \sum_{k=0}^{N-1} 1$$
$$= \frac{16}{N^2}\left(\frac{(N-1)^2}{2} + \frac{N-1}{2}\right) + \frac{4}{N}(N-1)$$
$$= 12 - \frac{12}{N}$$

The area under the graph is

$$\lim_{N\to\infty} L_N = 12.$$

The region under the curve over $[1, 3]$ is a trapezoid with base width 2 and heights 2 and 10. From this, we get that the area is $\frac{1}{2}(2)(2+10) = 12$, which agrees with the answer obtained from the limit of the left-endpoint approximations.

49. $\lim_{N\to\infty} M_N, \quad f(x) = x, \quad [0, 2]$

SOLUTION Let $f(x) = x$ on $[0, 2]$. Let $N > 0$ be an integer and set $a = 0$, $b = 2$, and $\Delta x = (b-a)/N = \frac{2}{N}$. Also, let $x_k^* = 0 + (k - \frac{1}{2})\Delta x = \frac{2k-1}{N}$, $k = 1, 2, \dots N$, be the midpoints of the N subintervals of $[0, 2]$. Then

$$M_N = \Delta x \sum_{k=1}^{N} f(x_k^*) = \frac{2}{N} \sum_{k=1}^{N} \frac{2k-1}{N} = \frac{2}{N^2} \sum_{k=1}^{N} (2k-1)$$
$$= \frac{2}{N^2}\left(2\sum_{k=1}^{N} k - N\right) = \frac{4}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) - \frac{2}{N} = 2.$$

The area under the curve over $[0, 2]$ is

$$\lim_{N\to\infty} M_N = 2.$$

The region under the curve over $[0, 2]$ is a triangle with base and height 2, and thus area 2, which agrees with the answer obtained from the limit of the midpoint approximations.

50. $\lim_{N\to\infty} M_N$, $f(x) = 12 - 4x$, $[2, 6]$

SOLUTION Let $f(x) = 12 - 4x$ on $[2, 6]$. Let $N > 0$ be an integer and set $a = 2$, $b = 6$, and $\Delta x = (b-a)/N = \frac{4}{N}$. Also, let $x_k^* = a + (k - \frac{1}{2})\Delta x = 2 + \frac{4k-2}{N}$, $k = 1, 2, \ldots N$, be the midpoints of the N subintervals of $[2, 6]$. Then

$$M_N = \Delta x \sum_{k=1}^{N} f(x_k^*) = \frac{4}{N}\sum_{k=1}^{N}\left(4 - \frac{16k-8}{N}\right)$$

$$= \frac{16}{N}\sum_{k=1}^{N} 1 - \frac{64}{N^2}\sum_{k=1}^{N} k + \frac{32}{N^2}\sum_{k=1}^{N} 1$$

$$= \frac{16}{N}(N) - \frac{64}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) + \frac{32}{N^2}(N) = -16.$$

The area under the curve over $[2, 6]$ is

$$\lim_{N\to\infty} M_N = -16.$$

The region under the curve over $[2, 6]$ consists of a triangle of base 1 and height 4 above the axis and a triangle of base 3 and height 12 below the axis. The area of this region is therefore

$$\frac{1}{2}(1)(4) - \frac{1}{2}(3)(12) = -16,$$

which agrees with the answer obtained from the limit of the midpoint approximations.

51. Show, for $f(x) = 3x^2 + 4x$ over $[0, 2]$, that

$$R_N = \frac{2}{N}\sum_{j=1}^{N}\left(\frac{24j^2}{N^2} + \frac{16j}{N}\right)$$

Then evaluate $\lim_{N\to\infty} R_N$.

SOLUTION Let $f(x) = 3x^2 + 4x$ on $[0, 2]$. Let N be a positive integer and set $a = 0$, $b = 2$, and $\Delta x = (b-a)/N = (2-0)/N = 2/N$. Also, let $x_j = a + j\Delta x = 2j/N$, $j = 1, 2, \ldots, N$ be the right endpoints of the N subintervals of $[0, 3]$. Then

$$R_N = \Delta x \sum_{j=1}^{N} f(x_j) = \frac{2}{N}\sum_{j=1}^{N}\left(3\left(\frac{2j}{N}\right)^2 + 4\frac{2j}{N}\right)$$

$$= \frac{2}{N}\sum_{j=1}^{N}\left(\frac{12j^2}{N^2} + \frac{8j}{N}\right)$$

Continuing, we find

$$R_N = \frac{24}{N^3}\sum_{j=1}^{N} j^2 + \frac{16}{N^2}\sum_{j=1}^{N} j$$

$$= \frac{24}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) + \frac{16}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right)$$

$$= 16 + \frac{20}{N} + \frac{4}{N^2}$$

Thus,

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(16 + \frac{20}{N} + \frac{4}{N^2}\right) = 16.$$

52. Show, for $f(x) = 3x^3 - x^2$ over $[1, 5]$, that

$$R_N = \frac{4}{N}\sum_{j=1}^{N}\left(\frac{192j^3}{N^3} + \frac{128j^2}{N^2} + \frac{28j}{N} + 2\right)$$

Then evaluate $\lim_{N\to\infty} R_N$.

SOLUTION Let $f(x) = 3x^3 - x^2$ on $[1, 5]$. Let N be a positive integer and set $a = 1$, $b = 5$, and $\Delta x = (b - a)/N = (5 - 1)/N = 4/N$. Also, let $x_j = a + j\Delta x = 1 + 4j/N$, $j = 1, 2, \ldots, N$ be the right endpoints of the N subintervals of $[1, 5]$. Then

$$\begin{aligned} f(x_j) &= 3\left(1 + \frac{4j}{N}\right)^3 - \left(1 + \frac{4j}{N}\right)^2 \\ &= 3\left(1 + \frac{12j}{N} + \frac{48j^2}{N^2} + \frac{64j^3}{N^3}\right) - \left(1 + \frac{8j}{N} + \frac{16j^2}{N^2}\right) \\ &= \frac{192j^3}{N^3} + \frac{128j^2}{N^2} + \frac{28j}{N} + 2. \end{aligned}$$

and

$$R_N = \sum_{j=1}^{N} f(x_j)\Delta x = \frac{4}{N}\sum_{j=1}^{N}\left(\frac{192j^3}{N^3} + \frac{128j^2}{N^2} + \frac{28j}{N} + 2\right).$$

Continuing, we find

$$\begin{aligned} R_N &= \frac{768}{N^4}\sum_{j=1}^{N} j^3 + \frac{512}{N^3}\sum_{j=1}^{N} j^2 + \frac{112}{N^2}\sum_{j=1}^{N} j + \frac{8}{N}\sum_{j=1}^{N} 1 \\ &= \frac{768}{N^4}\left(\frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{2}\right) + \frac{512}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) \\ &\quad + \frac{112}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) + \frac{8}{N}(N) \\ &= \frac{1280}{3} + \frac{696}{N} + \frac{832}{3N^2}. \end{aligned}$$

Thus,

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{1280}{3} + \frac{696}{N} + \frac{832}{3N^2}\right) = \frac{1280}{3}.$$

In Exercises 53–60, find a formula for R_N and compute the area under the graph as a limit.

53. $f(x) = x^2$, $[0, 1]$

SOLUTION Let $f(x) = x^2$ on the interval $[0, 1]$. Then $\Delta x = \frac{1 - 0}{N} = \frac{1}{N}$ and $a = 0$. Hence,

$$R_N = \Delta x\sum_{j=1}^{N} f(0 + j\Delta x) = \frac{1}{N}\sum_{j=1}^{N} j^2\frac{1}{N^2} = \frac{1}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) = \frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2}\right) = \frac{1}{3}.$$

54. $f(x) = x^2$, $[-1, 5]$

SOLUTION Let $f(x) = x^2$ on the interval $[-1, 5]$. Then $\Delta x = \frac{5 - (-1)}{N} = \frac{6}{N}$ and $a = -1$. Hence,

$$\begin{aligned} R_N &= \Delta x\sum_{j=1}^{N} f(-1 + j\Delta x) = \frac{6}{N}\sum_{j=1}^{N}\left(-1 + \frac{6j}{N}\right)^2 \\ &= \frac{6}{N}\sum_{j=1}^{N} 1 - \frac{72}{N^2}\sum_{j=1}^{N} j + \frac{216}{N^3}\sum_{j=1}^{N} j^2 \\ &= \frac{6}{N}(N) - \frac{72}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) + \frac{216}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) \\ &= 42 + \frac{72}{N} + \frac{36}{N^2} \end{aligned}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(42 + \frac{72}{N} + \frac{36}{N^2}\right) = 42.$$

55. $f(x) = 6x^2 - 4, \quad [2, 5]$

SOLUTION Let $f(x) = 6x^2 - 4$ on the interval $[2, 5]$. Then $\Delta x = \frac{5-2}{N} = \frac{3}{N}$ and $a = 2$. Hence,

$$\begin{aligned}
R_N &= \Delta x \sum_{j=1}^{N} f(2 + j\Delta x) = \frac{3}{N}\sum_{j=1}^{N}\left(6\left(2+\frac{3j}{N}\right)^2 - 4\right) = \frac{3}{N}\sum_{j=1}^{N}\left(20 + \frac{72j}{N} + \frac{54j^2}{N^2}\right) \\
&= 60 + \frac{216}{N^2}\sum_{j=1}^{N} j + \frac{162}{N^3}\sum_{j=1}^{N} j^2 \\
&= 60 + \frac{216}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) + \frac{162}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) \\
&= 222 + \frac{189}{N} + \frac{27}{N^2}
\end{aligned}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(222 + \frac{189}{N} + \frac{27}{N^2}\right) = 222.$$

56. $f(x) = x^2 + 7x, \quad [6, 11]$

SOLUTION Let $f(x) = x^2 + 7x$ on the interval $[6, 11]$. Then $\Delta x = \frac{11-6}{N} = \frac{5}{N}$ and $a = 6$. Hence,

$$\begin{aligned}
R_N &= \Delta x \sum_{j=1}^{N} f(6 + j\Delta x) = \frac{5}{N}\sum_{j=1}^{N}\left[\left(6+\frac{5j}{N}\right)^2 + 7\left(6+\frac{5j}{N}\right)\right] \\
&= \frac{5}{N}\sum_{j=1}^{N}\left(\frac{25j^2}{N^2} + \frac{95j}{N} + 78\right) \\
&= \frac{125}{N^3}\sum_{j=1}^{N} j^3 + \frac{475}{N^2}\sum_{j=1}^{N} j + \frac{390}{N}\sum_{j=1}^{N} 1 \\
&= \frac{125}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) + \frac{475}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) + 390 \\
&= \frac{4015}{6} + \frac{300}{N} + \frac{125}{6N^2}
\end{aligned}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{4015}{6} + \frac{300}{N} + \frac{125}{6N^2}\right) = \frac{4015}{6}.$$

57. $f(x) = x^3 - x, \quad [0, 2]$

SOLUTION Let $f(x) = x^3 - x$ on the interval $[0, 2]$. Then $\Delta x = \frac{2-0}{N} = \frac{2}{N}$ and $a = 0$. Hence,

$$\begin{aligned}
R_N &= \Delta x \sum_{j=1}^{N} f(0 + j\Delta x) = \frac{2}{N}\sum_{j=1}^{N}\left(\left(\frac{2j}{N}\right)^3 - \frac{2j}{N}\right) = \frac{2}{N}\sum_{j=1}^{N}\left(\frac{8j^3}{N^3} - \frac{2j}{N}\right) \\
&= \frac{16}{N^4}\sum_{j=1}^{N} j^3 - \frac{4}{N^2}\sum_{j=1}^{N} j \\
&= \frac{16}{N^4}\left(\frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{2}\right) - \frac{4}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) \\
&= 2 + \frac{6}{N} + \frac{8}{N^2}
\end{aligned}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(2+\frac{6}{N}+\frac{8}{N^2}\right)=2.$$

58. $f(x)=2x^3+x^2,\quad [-2,2]$

SOLUTION Let $f(x)=2x^3+x^2$ on the interval $[-2,2]$. Then $\Delta x=\dfrac{2-(-2)}{N}=\dfrac{4}{N}$ and $a=-2$. Hence,

$$\begin{aligned}R_N &= \Delta x\sum_{j=1}^{N} f(-2+j\Delta x)=\frac{4}{N}\sum_{j=1}^{N}\left[2\left(-2+\frac{4j}{N}\right)^3+\left(-2+\frac{4j}{N}\right)^2\right]\\ &=\frac{4}{N}\sum_{j=1}^{N}\left(\frac{128j^3}{N^3}-\frac{176j^2}{N^2}+\frac{80j}{N}-12\right)\\ &=\frac{512}{N^4}\left(\frac{N^4}{4}+\frac{N^3}{2}+\frac{N^2}{4}\right)-\frac{704}{N^3}\left(\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}\right)+\frac{320}{N^2}\left(\frac{N^2}{2}+\frac{N}{2}\right)-48\\ &=\frac{16}{3}+\frac{64}{N}+\frac{32}{3N^2}\end{aligned}$$

and

$$\lim_{N\to\infty} R_N=\lim_{N\to\infty}\left(\frac{16}{3}+\frac{64}{N}+\frac{32}{3N^2}\right)=\frac{16}{3}.$$

59. $f(x)=2x+1,\quad [a,b]\quad (a,b$ constants with $a<b)$

SOLUTION Let $f(x)=2x+1$ on the interval $[a,b]$. Then $\Delta x=\dfrac{b-a}{N}$. Hence,

$$\begin{aligned}R_N &= \Delta x\sum_{j=1}^{N} f(a+j\Delta x)=\frac{(b-a)}{N}\sum_{j=1}^{N}\left(2\left(a+j\frac{(b-a)}{N}\right)+1\right)\\ &=\frac{(b-a)}{N}(2a+1)\sum_{j=1}^{N}1+\frac{2(b-a)^2}{N^2}\sum_{j=1}^{N}j\\ &=\frac{(b-a)}{N}(2a+1)N+\frac{2(b-a)^2}{N^2}\left(\frac{N^2}{2}+\frac{N}{2}\right)\\ &=(b-a)(2a+1)+(b-a)^2+\frac{(b-a)^2}{N}\end{aligned}$$

and

$$\begin{aligned}\lim_{N\to\infty} R_N &=\lim_{N\to\infty}\left((b-a)(2a+1)+(b-a)^2+\frac{(b-a)^2}{N}\right)\\ &=(b-a)(2a+1)+(b-a)^2=(b^2+b)-(a^2+a).\end{aligned}$$

60. $f(x)=x^2,\quad [a,b]\quad (a,b$ constants with $a<b)$

SOLUTION Let $f(x)=x^2$ on the interval $[a,b]$. Then $\Delta x=\dfrac{b-a}{N}$. Hence,

$$\begin{aligned}R_N &= \Delta x\sum_{j=1}^{N} f(a+j\Delta x)=\frac{(b-a)}{N}\sum_{j=1}^{N}\left(a^2+2aj\frac{(b-a)}{N}+j^2\frac{(b-a)^2}{N^2}\right)\\ &=\frac{a^2(b-a)}{N}\sum_{j=1}^{N}1+\frac{2a(b-a)^2}{N^2}\sum_{j=1}^{N}j+\frac{(b-a)^3}{N^3}\sum_{j=1}^{N}j^2\\ &=\frac{a^2(b-a)}{N}N+\frac{2a(b-a)^2}{N^2}\left(\frac{N^2}{2}+\frac{N}{2}\right)+\frac{(b-a)^3}{N^3}\left(\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}\right)\\ &=a^2(b-a)+a(b-a)^2+\frac{a(b-a)^2}{N}+\frac{(b-a)^3}{3}+\frac{(b-a)^3}{2N}+\frac{(b-a)^3}{6N^2}\end{aligned}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(a^2(b-a)+a(b-a)^2+\frac{a(b-a)^2}{N}+\frac{(b-a)^3}{3}+\frac{(b-a)^3}{2N}+\frac{(b-a)^3}{6N^2}\right)$$

$$= a^2(b-a)+a(b-a)^2+\frac{(b-a)^3}{3} = \frac{1}{3}b^3-\frac{1}{3}a^3.$$

In Exercises 61–64, describe the area represented by the limits.

61. $\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\left(\frac{j}{N}\right)^4$

SOLUTION The limit

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\left(\frac{j}{N}\right)^4$$

represents the area between the graph of $f(x)=x^4$ and the x-axis over the interval [0, 1].

62. $\lim_{N\to\infty}\frac{3}{N}\sum_{j=1}^{N}\left(2+\frac{3j}{N}\right)^4$

SOLUTION The limit

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\frac{3}{N}\sum_{j=1}^{N}\left(2+j\cdot\frac{3}{N}\right)^4$$

represents the area between the graph of $f(x)=x^4$ and the x-axis over the interval [2, 5].

63. $\lim_{N\to\infty}\frac{5}{N}\sum_{j=0}^{N-1}e^{-2+5j/N}$

SOLUTION The limit

$$\lim_{N\to\infty} L_N = \lim_{N\to\infty}\frac{5}{N}\sum_{j=0}^{N-1}e^{-2+5j/N}$$

represents the area between the graph of $y=e^x$ and the x-axis over the interval $[-2, 3]$.

64. $\lim_{N\to\infty}\frac{\pi}{2N}\sum_{j=1}^{N}\sin\left(\frac{\pi}{3}-\frac{\pi}{4N}+\frac{j\pi}{2N}\right)$

SOLUTION The limit

$$\lim_{N\to\infty}\frac{\pi}{2N}\sum_{j=1}^{N}\sin\left(\frac{\pi}{3}-\frac{\pi}{4N}+\frac{j\pi}{2N}\right)$$

represents the area between the graph of $y=\sin x$ and the x-axis over the interval $[\frac{\pi}{3}, \frac{5\pi}{6}]$.

In Exercises 65–70, express the area under the graph as a limit using the approximation indicated (in summation notation), but do not evaluate.

65. R_N, $f(x)=\sin x$ over $[0,\pi]$

SOLUTION Let $f(x)=\sin x$ over $[0,\pi]$ and set $a=0$, $b=\pi$, and $\Delta x=(b-a)/N=\pi/N$. Then

$$R_N=\Delta x\sum_{k=1}^{N}f(x_k)=\frac{\pi}{N}\sum_{k=1}^{N}\sin\left(\frac{k\pi}{N}\right).$$

Hence

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\frac{\pi}{N}\sum_{k=1}^{N}\sin\left(\frac{k\pi}{N}\right)$$

is the area between the graph of $f(x)=\sin x$ and the x-axis over $[0,\pi]$.

66. R_N, $f(x) = x^{-1}$ over $[1, 7]$

SOLUTION Let $f(x) = x^{-1}$ over the interval $[1, 7]$. Then $\Delta x = \dfrac{7-1}{N} = \dfrac{6}{N}$ and $a = 1$. Hence,

$$R_N = \Delta x \sum_{j=1}^{N} f(1 + j\Delta x) = \frac{6}{N}\sum_{j=1}^{N}\left(1 + j\frac{6}{N}\right)^{-1}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\frac{6}{N}\sum_{j=1}^{N}\left(1 + j\frac{6}{N}\right)^{-1}$$

is the area between the graph of $f(x) = x^{-1}$ and the x-axis over $[1, 7]$.

67. L_N, $f(x) = \sqrt{2x+1}$ over $[7, 11]$

SOLUTION Let $f(x) = \sqrt{2x+1}$ over the interval $[7, 11]$. Then $\Delta x = \dfrac{11-7}{N} = \dfrac{4}{N}$ and $a = 7$. Hence,

$$L_N = \Delta x \sum_{j=0}^{N-1} f(7 + j\Delta x) = \frac{4}{N}\sum_{j=0}^{N-1}\sqrt{2(7 + j\frac{4}{N}) + 1}$$

and

$$\lim_{N\to\infty} L_N = \lim_{N\to\infty}\frac{4}{N}\sum_{j=0}^{N-1}\sqrt{15 + \frac{8j}{N}}$$

is the area between the graph of $f(x) = \sqrt{2x+1}$ and the x-axis over $[7, 11]$.

68. L_N, $f(x) = \cos x$ over $\left[\frac{\pi}{8}, \frac{\pi}{4}\right]$

SOLUTION Let $f(x) = \cos x$ over the interval $\left[\frac{\pi}{8}, \frac{\pi}{4}\right]$. Then $\Delta x = \dfrac{\frac{\pi}{4} - \frac{\pi}{8}}{N} = \dfrac{\frac{\pi}{8}}{N} = \dfrac{\pi}{8N}$ and $a = \frac{\pi}{8}$, Hence:

$$L_N = \Delta x \sum_{j=0}^{N-1} f\left(\frac{\pi}{8} + j\Delta x\right) = \frac{\pi}{8N}\sum_{j=0}^{N-1}\cos\left(\frac{\pi}{8} + j\frac{\pi}{8N}\right)$$

and

$$\lim_{N\to\infty} L_N = \lim_{N\to\infty}\frac{\pi}{8N}\sum_{j=0}^{N-1}\cos\left(\frac{\pi}{8} + j\frac{\pi}{8N}\right)$$

is the area between the graph of $f(x) = \cos x$ and the x-axis over $[\frac{\pi}{8}, \frac{\pi}{4}]$.

69. M_N, $f(x) = \tan x$ over $\left[\frac{1}{2}, 1\right]$

SOLUTION Let $f(x) = \tan x$ over the interval $[\frac{1}{2}, 1]$. Then $\Delta x = \frac{1-\frac{1}{2}}{N} = \frac{1}{2N}$ and $a = \frac{1}{2}$. Hence

$$M_N = \Delta x \sum_{j=1}^{N} f\left(\frac{1}{2} + \left(j - \frac{1}{2}\right)\Delta x\right) = \frac{1}{2N}\sum_{j=1}^{N}\tan\left(\frac{1}{2} + \frac{1}{2N}\left(j - \frac{1}{2}\right)\right)$$

and so

$$\lim_{N\to\infty} M_N = \lim_{N\to\infty}\frac{1}{2N}\sum_{j=1}^{N}\tan\left(\frac{1}{2} + \frac{1}{2N}\left(j - \frac{1}{2}\right)\right)$$

is the area between the graph of $f(x) = \tan x$ and the x-axis over $[\frac{1}{2}, 1]$.

70. M_N, $f(x) = x^{-2}$ over $[3, 5]$

SOLUTION Let $f(x) = x^{-2}$ over the interval $[3, 5]$. Then $\Delta x = \frac{5-3}{N} = \frac{2}{N}$ and $a = 3$. Hence

$$M_N = \Delta x \sum_{j=1}^{N} f\left(3 + \left(j - \frac{1}{2}\right)\Delta x\right) = \frac{2}{N}\sum_{j=1}^{N}\left(3 + \frac{2}{N}\left(j - \frac{1}{2}\right)\right)^{-2}$$

and so

$$\lim_{N\to\infty} M_N = \lim_{N\to\infty} \frac{2}{N}\sum_{j=1}^{N}\left(3+\frac{2}{N}\left(j-\frac{1}{2}\right)\right)^{-2}$$

is the area between the graph of $f(x) = x^{-2}$ and the x-axis over [3, 5].

71. Evaluate $\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N}\sqrt{1-\left(\frac{j}{N}\right)^2}$ by interpreting it as the area of part of a familiar geometric figure.

SOLUTION The limit

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N}\sqrt{1-\left(\frac{j}{N}\right)^2}$$

represents the area between the graph of $y = f(x) = \sqrt{1-x^2}$ and the x-axis over the interval [0, 1]. This is the portion of the circular disk $x^2 + y^2 \le 1$ that lies in the first quadrant. Accordingly, its area is $\frac{1}{4}\pi\,(1)^2 = \frac{\pi}{4}$.

In Exercises 72–74, let $f(x) = x^2$ and let R_N, L_N, and M_N be the approximations for the interval [0, 1].

72. Show that $R_N = \frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2}$. Interpret the quantity $\frac{1}{2N} + \frac{1}{6N^2}$ as the area of a region.

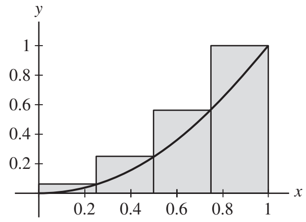
SOLUTION Let $f(x) = x^2$ on [0, 1]. Let $N > 0$ be an integer and set $a = 0$, $b = 1$ and $\Delta x = \frac{1-0}{N} = \frac{1}{N}$. Then

$$R_N = \Delta x \sum_{j=1}^{N} f(0 + j\Delta x) = \frac{1}{N}\sum_{j=1}^{N} j^2 \frac{1}{N^2} = \frac{1}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) = \frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2}.$$

The quantity

$$\frac{1}{2N} + \frac{6}{N^2} \quad \text{in} \quad R_N = \frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2}$$

represents the collective area of the parts of the rectangles that lie above the graph of $f(x)$. It is the error between R_N and the true area $A = \frac{1}{3}$.



73. Show that

$$L_N = \frac{1}{3} - \frac{1}{2N} + \frac{1}{6N^2}, \qquad M_N = \frac{1}{3} - \frac{1}{12N^2}$$

Then rank the three approximations R_N, L_N, and M_N in order of increasing accuracy (use Exercise 72).

SOLUTION Let $f(x) = x^2$ on [0, 1]. Let N be a positive integer and set $a = 0$, $b = 1$, and $\Delta x = (b-a)/N = 1/N$. Let $x_k = a + k\Delta x = k/N$, $k = 0, 1, \ldots, N$ and let $x_k^* = a + (k+\frac{1}{2})\Delta x = (k+\frac{1}{2})/N$, $k = 0, 1, \ldots, N-1$. Then

$$L_N = \Delta x \sum_{k=0}^{N-1} f(x_k) = \frac{1}{N}\sum_{k=0}^{N-1}\left(\frac{k}{N}\right)^2 = \frac{1}{N^3}\sum_{k=1}^{N-1} k^2$$

$$= \frac{1}{N^3}\left(\frac{(N-1)^3}{3} + \frac{(N-1)^2}{2} + \frac{N-1}{6}\right) = \frac{1}{3} - \frac{1}{2N} + \frac{1}{6N^2}$$

$$M_N = \Delta x \sum_{k=0}^{N-1} f(x_k^*) = \frac{1}{N}\sum_{k=0}^{N-1}\left(\frac{k+\frac{1}{2}}{N}\right)^2 = \frac{1}{N^3}\sum_{k=0}^{N-1}\left(k^2+k+\frac{1}{4}\right)$$

$$= \frac{1}{N^3}\left(\left(\sum_{k=1}^{N-1}k^2\right)+\left(\sum_{k=1}^{N-1}k\right)+\frac{1}{4}\left(\sum_{k=0}^{N-1}1\right)\right)$$

$$= \frac{1}{N^3}\left(\left(\frac{(N-1)^3}{3}+\frac{(N-1)^2}{2}+\frac{N-1}{6}\right)+\left(\frac{(N-1)^2}{2}+\frac{N-1}{2}\right)+\frac{1}{4}N\right) = \frac{1}{3}-\frac{1}{12N^2}$$

The error of R_N is given by $\frac{1}{2N}+\frac{1}{6N^2}$, the error of L_N is given by $-\frac{1}{2N}+\frac{1}{6N^2}$ and the error of M_N is given by $-\frac{1}{12N^2}$. Of the three approximations, R_N is the least accurate, then L_N and finally M_N is the most accurate.

74. For each of R_N, L_N, and M_N, find the smallest integer N for which the error is less than 0.001.

SOLUTION

- For R_N, the error is less than 0.001 when:

$$\frac{1}{2N}+\frac{1}{6N^2} < 0.001.$$

We find an adequate solution in N:

$$\frac{1}{2N}+\frac{1}{6N^2} < 0.001$$
$$3N+1 < 0.006(N^2)$$
$$0 < 0.006N^2-3N-1,$$

in particular, if $N > \frac{3+\sqrt{9.024}}{0.012} = 500.333$. Hence R_{501} is within 0.001 of A.

- For L_N, the error is less than 0.001 if

$$\left|-\frac{1}{2N}+\frac{1}{6N^2}\right| < 0.001.$$

We solve this equation for N:

$$\left|\frac{1}{2N}-\frac{1}{6N^2}\right| < 0.001$$
$$\left|\frac{3N-1}{6N^2}\right| < 0.001$$
$$3N-1 < 0.006N^2$$
$$0 < 0.006N^2-3N+1,$$

which is satisfied if $N > \frac{3+\sqrt{9-0.024}}{0.012} = 499.666$. Therefore, L_{500} is within 0.001 units of A.

- For M_N, the error is given by $-\frac{1}{12N^2}$, so the error is less than 0.001 if

$$\frac{1}{12N^2} < 0.001$$
$$1000 < 12N^2$$
$$9.13 < N$$

Therefore, M_{10} is within 0.001 units of the correct answer.

In Exercises 75–80, use the Graphical Insight on page 291 to obtain bounds on the area.

75. Let A be the area under $f(x)=\sqrt{x}$ over [0, 1]. Prove that $0.51 \le A \le 0.77$ by computing R_4 and L_4. Explain your reasoning.

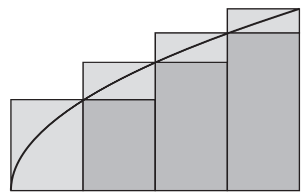
SOLUTION For $n=4$, $\Delta x = \frac{1-0}{4} = \frac{1}{4}$ and $\{x_i\}_{i=0}^{4} = \{0+i\Delta x\} = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. Therefore,

$$R_4 = \Delta x\sum_{i=1}^{4} f(x_i) = \frac{1}{4}\left(\frac{1}{2}+\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}+1\right) \approx 0.768$$

$$L_4 = \Delta x \sum_{i=0}^{3} f(x_i) = \frac{1}{4}\left(0 + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\right) \approx 0.518.$$

In the plot below, you can see the rectangles whose area is represented by L_4 under the graph and the top of those whose area is represented by R_4 above the graph. The area A under the curve is somewhere between L_4 and R_4, so

$$0.518 \le A \le 0.768.$$



L_4, R_4 and the graph of $f(x)$.

76. Use R_5 and L_5 to show that the area A under $y = x^{-2}$ over [10, 13] satisfies $0.0218 \le A \le 0.0244$.

SOLUTION Let $f(x) = x^{-2}$ over the interval [10, 13]. Because f is a decreasing function over this interval, it follows that $R_N \le A \le L_N$ for all N. Taking $N = 5$, we have $\Delta x = 3/5$ and

$$R_5 = \frac{3}{5}\left(\frac{1}{10.6^2} + \frac{1}{11.2^2} + \frac{1}{11.8^2} + \frac{1}{12.4^2} + \frac{1}{13^2}\right) = 0.021885.$$

Moreover,

$$L_5 = \frac{3}{5}\left(\frac{1}{10^2} + \frac{1}{10.6^2} + \frac{1}{11.2^2} + \frac{1}{11.8^2} + \frac{1}{12.4^2}\right) = 0.0243344.$$

Thus,

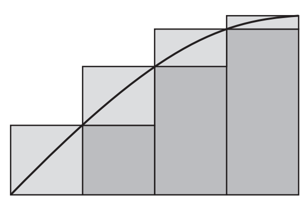
$$0.0218 < R_5 \le A \le L_5 < 0.0244.$$

77. Use R_4 and L_4 to show that the area A under the graph of $y = \sin x$ over $\left[0, \frac{\pi}{2}\right]$ satisfies $0.79 \le A \le 1.19$.

SOLUTION Let $f(x) = \sin x$. $f(x)$ is increasing over the interval $[0, \pi/2]$, so the Insight on page 291 applies, which indicates that $L_4 \le A \le R_4$. For $n = 4$, $\Delta x = \frac{\pi/2 - 0}{4} = \frac{\pi}{8}$ and $\{x_i\}_{i=0}^{4} = \{0 + i\Delta x\}_{i=0}^{4} = \{0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}\}$. From this,

$$L_4 = \frac{\pi}{8}\sum_{i=0}^{3} f(x_i) \approx 0.79, \qquad R_4 = \frac{\pi}{8}\sum_{i=1}^{4} f(x_i) \approx 1.18.$$

Hence A is between 0.79 and 1.19.



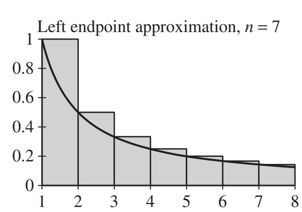
Left and Right endpoint approximations to A.

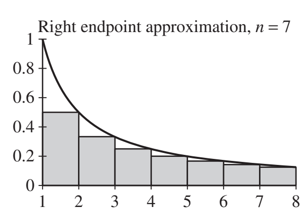
78. Show that the area A under $f(x) = x^{-1}$ over [1, 8] satisfies

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} \le A \le 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7}$$

SOLUTION Let $f(x) = x^{-1}$, $1 \le x \le 8$. Since f is decreasing, the left endpoint approximation L_7 overestimates the true area between the graph of f and the x-axis, whereas the right endpoint approximation R_7 underestimates it. Accordingly,

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = R_7 < A < L_7 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$$





79. CAS Show that the area A under $y = x^{1/4}$ over [0, 1] satisfies $L_N \le A \le R_N$ for all N. Use a computer algebra system to calculate L_N and R_N for $N = 100$ and 200, and determine A to two decimal places.

SOLUTION On [0, 1], $f(x) = x^{1/4}$ is an increasing function; therefore, $L_N \le A \le R_N$ for all N. We find

$$L_{100} = 0.793988 \quad \text{and} \quad R_{100} = 0.80399,$$

while

$$L_{200} = 0.797074 \quad \text{and} \quad R_{200} = 0.802075.$$

Thus, $A = 0.80$ to two decimal places.

80. CAS Show that the area A under $y = 4/(x^2 + 1)$ over [0, 1] satisfies $R_N \le A \le L_N$ for all N. Determine A to at least three decimal places using a computer algebra system. Can you guess the exact value of A?

SOLUTION On [0, 1], the function $f(x) = 4/(x^2 + 1)$ is decreasing, so $R_N \le A \le L_N$ for all N. From the values in the table below, we find $A = 3.142$ to three decimal places. It appears that the exact value of A is π.

N	R_N	L_N
10	3.03993	3.23992
100	3.13158	3.15158
1000	3.14059	3.14259
10000	3.14149	3.14169
100000	3.14158	3.14160

81. In this exercise, we evaluate the area A under the graph of $y = e^x$ over [0, 1] [Figure 19(A)] using the formula for a geometric sum (valid for $r \ne 1$):

$$1 + r + r^2 + \cdots + r^{N-1} = \sum_{j=0}^{N-1} r^j = \frac{r^N - 1}{r - 1} \tag{8}$$

(a) Show that $L_N = \dfrac{1}{N} \displaystyle\sum_{j=0}^{N-1} e^{j/N}$.

(b) Apply Eq. (8) with $r = e^{1/N}$ to prove $L_N = \dfrac{e - 1}{N(e^{1/N} - 1)}$.

(c) Compute $A = \lim_{N\to\infty} L_N$ using L'Hôpital's Rule.

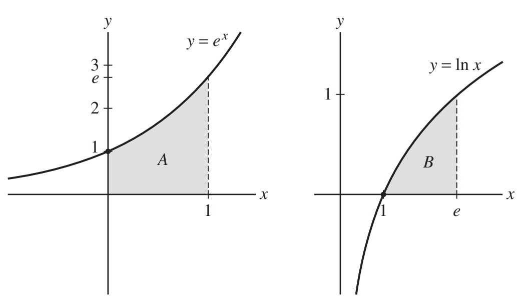


FIGURE 19

SOLUTION

(a) Let $f(x) = e^x$ on [0, 1]. With $n = N$, $\Delta x = (1 - 0)/N = 1/N$ and

$$x_j = a + j\Delta x = \frac{j}{N}$$

for $j = 0, 1, 2, \ldots, N$. Therefore,

$$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j) = \frac{1}{N} \sum_{j=0}^{N-1} e^{j/N}.$$

(b) Applying Eq. (8) with $r = e^{1/N}$, we have

$$L_N = \frac{1}{N} \frac{(e^{1/N})^N - 1}{e^{1/N} - 1} = \frac{e - 1}{N(e^{1/N} - 1)}.$$

Therefore,

$$A = \lim_{N\to\infty} L_N = (e-1) \lim_{N\to\infty} \frac{1}{N(e^{1/N}-1)}.$$

(c) Using L'Hôpital's Rule,

$$A = (e-1) \lim_{N\to\infty} \frac{N^{-1}}{e^{1/N}-1} = (e-1) \lim_{N\to\infty} \frac{-N^{-2}}{-N^{-2}e^{1/N}} = (e-1) \lim_{N\to\infty} e^{-1/N} = e-1.$$

82. Use the result of Exercise 81 to show that the area B under the graph of $f(x) = \ln x$ over $[1, e]$ is equal to 1. *Hint:* Relate B in Figure 19(B) to the area A computed in Exercise 81.

SOLUTION Because $y = \ln x$ and $y = e^x$ are inverse functions, we note that if the area B is reflected across the line $y = x$ and then combined with the area A, we create a rectangle of width 1 and height e. The area of this rectangle is therefore e, and it follows that the area B is equal to e minus the area A. Using the result of Exercise 81, the area B is equal to

$$e - (e-1) = 1.$$

Further Insights and Challenges

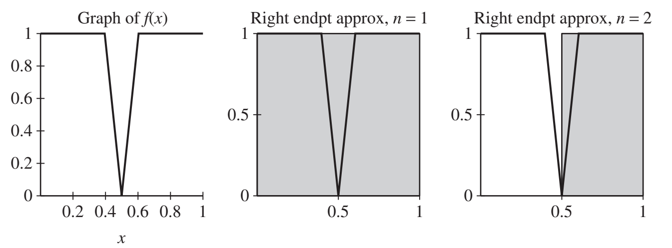
83. Although the accuracy of R_N generally improves as N increases, this need not be true for small values of N. Draw the graph of a positive continuous function $f(x)$ on an interval such that R_1 is closer than R_2 to the exact area under the graph. Can such a function be monotonic?

SOLUTION Let δ be a small positive number less than $\frac{1}{4}$. (In the figures below, $\delta = \frac{1}{10}$. But imagine δ being *very* tiny.) Define $f(x)$ on $[0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2} - \delta \\ \frac{1}{2\delta} - \frac{x}{\delta} & \text{if } \frac{1}{2} - \delta \le x < \frac{1}{2} \\ \frac{x}{\delta} - \frac{1}{2\delta} & \text{if } \frac{1}{2} \le x < \frac{1}{2} + \delta \\ 1 & \text{if } \frac{1}{2} + \delta \le x \le 1 \end{cases}$$

Then f is continuous on $[0, 1]$. (Again, just look at the figures.)

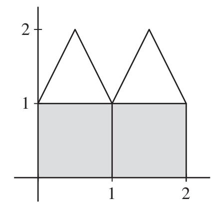
- The exact area between f and the x-axis is $A = 1 - \frac{1}{2}bh = 1 - \frac{1}{2}(2\delta)(1) = 1 - \delta$. (For $\delta = \frac{1}{10}$, we have $A = \frac{9}{10}$.)
- With $R_1 = 1$, the absolute error is $|E_1| = |R_1 - A| = |1 - (1-\delta)| = \delta$. (For $\delta = \frac{1}{10}$, this absolute error is $|E_1| = \frac{1}{10}$.)
- With $R_2 = \frac{1}{2}$, the absolute error is $|E_2| = |R_2 - A| = \left|\frac{1}{2} - (1-\delta)\right| = \left|\delta - \frac{1}{2}\right| = \frac{1}{2} - \delta$. (For $\delta = \frac{1}{10}$, we have $|E_2| = \frac{2}{5}$.)
- Accordingly, R_1 is closer to the exact area A than is R_2. Indeed, the tinier δ is, the more dramatic the effect.
- For a monotonic function, this phenomenon cannot occur. Successive approximations from either side get progressively more accurate.



84. Draw the graph of a positive continuous function on an interval such that R_2 and L_2 are both smaller than the exact area under the graph. Can such a function be monotonic?

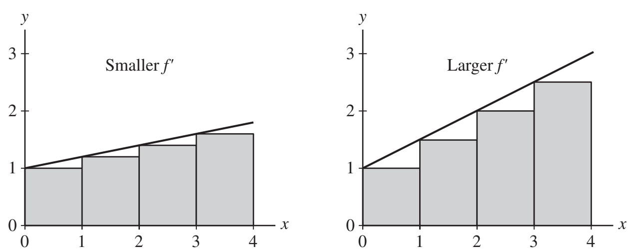
SOLUTION In the plot below, the area under the saw-tooth function $f(x)$ is 3, whereas $L_2 = R_2 = 2$. Thus L_2 and R_2 are both smaller than the exact area. Such a function cannot be monotonic; if $f(x)$ is increasing, then L_N underestimates and R_N overestimates the area for all N, and, if $f(x)$ is decreasing, then L_N overestimates and R_N underestimates the area for all N.

Left/right-endpoint approximation, $n = 2$



85. Explain graphically: *The endpoint approximations are less accurate when $f'(x)$ is large.*

SOLUTION When f' is large, the graph of f is steeper and hence there is more gap between f and L_N or R_N. Recall that the top line segments of the rectangles involved in an endpoint approximation constitute a piecewise constant function. If f' is large, then f is increasing more rapidly and hence is less like a constant function.



86. Prove that for any function $f(x)$ on $[a, b]$,

$$R_N - L_N = \frac{b-a}{N}(f(b) - f(a)) \qquad \boxed{9}$$

SOLUTION For any f (continuous or not) on $I = [a, b]$, partition I into N equal subintervals. Let $\Delta x = (b-a)/N$ and set $x_k = a + k\Delta x$, $k = 0, 1, \ldots N$. Then we have the following approximations to the area between the graph of f and the x-axis: the left endpoint approximation $L_N = \Delta x \sum_{k=0}^{N-1} f(x_k)$ and right endpoint approximation $R_N = \Delta x \sum_{k=1}^{N} f(x_k)$. Accordingly,

$$\begin{aligned} R_N - L_N &= \left(\Delta x \sum_{k=1}^{N} f(x_k)\right) - \left(\Delta x \sum_{k=0}^{N-1} f(x_k)\right) \\ &= \Delta x \left(f(x_N) + \left(\sum_{k=1}^{N-1} f(x_k)\right) - f(x_0) - \left(\sum_{k=1}^{N-1} f(x_k)\right)\right) \\ &= \Delta x\, (f(x_N) - f(x_0)) = \frac{b-a}{N}(f(b) - f(a)). \end{aligned}$$

In other words, $R_N - L_N = \dfrac{b-a}{N}(f(b) - f(a))$.

87. In this exercise, we prove that $\lim_{N\to\infty} R_N$ and $\lim_{N\to\infty} L_N$ exist and are equal if $f(x)$ is increasing [the case of $f(x)$ decreasing is similar]. We use the concept of a least upper bound discussed in Appendix B.

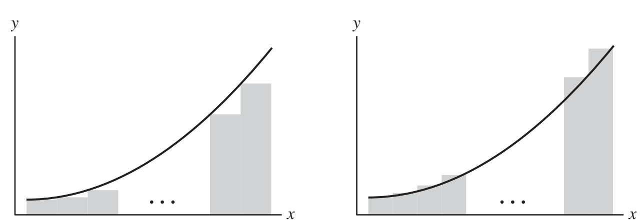
(a) Explain with a graph why $L_N \le R_M$ for all $N, M \ge 1$.

(b) By (a), the sequence $\{L_N\}$ is bounded, so it has a least upper bound L. By definition, L is the smallest number such that $L_N \le L$ for all N. Show that $L \le R_M$ for all M.

(c) According to (b), $L_N \le L \le R_N$ for all N. Use Eq. (9) to show that $\lim_{N\to\infty} L_N = L$ and $\lim_{N\to\infty} R_N = L$.

SOLUTION

(a) Let $f(x)$ be positive and increasing, and let N and M be positive integers. From the figure below at the left, we see that L_N underestimates the area under the graph of $y = f(x)$, while from the figure below at the right, we see that R_M overestimates the area under the graph. Thus, for all $N, M \ge 1$, $L_N \le R_M$.



(b) Because the sequence $\{L_N\}$ is bounded above by R_M for any M, each R_M is an upper bound for the sequence. Furthermore, the sequence $\{L_N\}$ must have a least upper bound, call it L. By definition, the least upper bound must be no greater than any other upper bound; consequently, $L \le R_M$ for all M.

(c) Since $L_N \le L \le R_N$, $R_N - L \le R_N - L_N$, so $|R_N - L| \le |R_N - L_N|$. From this,

$$\lim_{N\to\infty} |R_N - L| \le \lim_{N\to\infty} |R_N - L_N|.$$

By Eq. (9),

$$\lim_{N\to\infty} |R_N - L_N| = \lim_{N\to\infty} \frac{1}{N}|(b-a)(f(b) - f(a))| = 0,$$

so $\lim\limits_{N\to\infty} |R_N - L| \le |R_N - L_N| = 0$, hence $\lim\limits_{N\to\infty} R_N = L$.

Similarly, $|L_N - L| = L - L_N \le R_N - L_N$, so

$$|L_N - L| \le |R_N - L_N| = \frac{(b-a)}{N}(f(b) - f(a)).$$

This gives us that

$$\lim_{N\to\infty} |L_N - L| \le \lim_{N\to\infty} \frac{1}{N}|(b-a)(f(b) - f(a))| = 0,$$

so $\lim\limits_{N\to\infty} L_N = L$.

This proves $\lim\limits_{N\to\infty} L_N = \lim\limits_{N\to\infty} R_N = L$.

88. Use Eq. (9) to show that if $f(x)$ is positive and monotonic, then the area A under its graph over $[a, b]$ satisfies

$$|R_N - A| \le \frac{b-a}{N}|f(b) - f(a)| \qquad \boxed{10}$$

SOLUTION Let $f(x)$ be continuous, positive, and monotonic on $[a, b]$. Let A be the area between the graph of f and the x-axis over $[a, b]$. For specificity, say f is increasing. (The case for f decreasing on $[a, b]$ is similar.) As noted in the text, we have $L_N \le A \le R_N$. By Exercise 86 and the fact that A lies between L_N and R_N, we therefore have

$$0 \le R_N - A \le R_N - L_N = \frac{b-a}{N}(f(b) - f(a)).$$

Hence

$$|R_N - A| \le \frac{b-a}{N}(f(b) - f(a)) = \frac{b-a}{N}|f(b) - f(a)|,$$

where $f(b) - f(a) = |f(b) - f(a)|$ because f is increasing on $[a, b]$.

In Exercises 89 and 90, use Eq. (10) to find a value of N such that $|R_N - A| < 10^{-4}$ for the given function and interval.

89. $f(x) = \sqrt{x}$, $[1, 4]$

SOLUTION Let $f(x) = \sqrt{x}$ on $[1, 4]$. Then $b = 4$, $a = 1$, and

$$|R_N - A| \le \frac{4-1}{N}(f(4) - f(1)) = \frac{3}{N}(2-1) = \frac{3}{N}.$$

We need $\frac{3}{N} < 10^{-4}$, which gives $N > 30{,}000$. Thus $|R_{30{,}001} - A| < 10^{-4}$ for $f(x) = \sqrt{x}$ on $[1, 4]$.

90. $f(x) = \sqrt{9 - x^2}$, $[0, 3]$

SOLUTION Let $f(x) = \sqrt{9 - x^2}$ on $[0, 3]$. Then $b = 3$, $a = 0$, and

$$|R_N - A| \le \frac{b-a}{N}|f(b) - f(a)| = \frac{3}{N}(3) = \frac{9}{N}.$$

We need $\frac{9}{N} < 10^{-4}$, which gives $N > 90{,}000$. Thus $|R_{90{,}001} - A| < 10^{-4}$ for $f(x) = \sqrt{9 - x^2}$ on $[0, 3]$.

91. Prove that if $f(x)$ is positive and monotonic, then M_N lies between R_N and L_N and is closer to the actual area under the graph than both R_N and L_N. *Hint:* In the case that $f(x)$ is increasing, Figure 20 shows that the part of the error in R_N due to the ith rectangle is the sum of the areas $A + B + D$, and for M_N it is $|B - E|$. On the other hand, $A \ge E$.

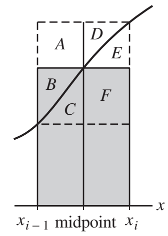


FIGURE 20

SOLUTION Suppose $f(x)$ is monotonic increasing on the interval $[a, b]$, $\Delta x = \frac{b-a}{N}$,

$$\{x_k\}_{k=0}^{N} = \{a, a + \Delta x, a + 2\Delta x, \ldots, a + (N-1)\Delta x, b\}$$

and

$$\left\{x_k^*\right\}_{k=0}^{N-1} = \left\{\frac{a + (a + \Delta x)}{2}, \frac{(a + \Delta x) + (a + 2\Delta x)}{2}, \ldots, \frac{(a + (N-1)\Delta x) + b}{2}\right\}.$$

Note that $x_i < x_i^* < x_{i+1}$ implies $f(x_i) < f(x_i^*) < f(x_{i+1})$ for all $0 \le i < N$ because $f(x)$ is monotone increasing. Then

$$\left(L_N = \frac{b-a}{N}\sum_{k=0}^{N-1} f(x_k)\right) < \left(M_N = \frac{b-a}{N}\sum_{k=0}^{N-1} f(x_k^*)\right) < \left(R_N = \frac{b-a}{N}\sum_{k=1}^{N} f(x_k)\right)$$

Similarly, if $f(x)$ is monotone decreasing,

$$\left(L_N = \frac{b-a}{N}\sum_{k=0}^{N-1} f(x_k)\right) > \left(M_N = \frac{b-a}{N}\sum_{k=0}^{N-1} f(x_k^*)\right) > \left(R_N = \frac{b-a}{N}\sum_{k=1}^{N} f(x_k)\right)$$

Thus, if $f(x)$ is monotonic, then M_N always lies in between R_N and L_N.

Now, as in Figure 20, consider the typical subinterval $[x_{i-1}, x_i]$ and its midpoint x_i^*. We let A, B, C, D, E, and F be the areas as shown in Figure 20. Note that, by the fact that x_i^* is the midpoint of the interval, $A = D + E$ and $F = B + C$. Let E_R represent the right endpoint approximation error ($= A + B + D$), let E_L represent the left endpoint approximation error ($= C + F + E$) and let E_M represent the midpoint approximation error ($= |B - E|$).

- If $B > E$, then $E_M = B - E$. In this case,

  $$E_R - E_M = A + B + D - (B - E) = A + D + E > 0,$$

  so $E_R > E_M$, while

  $$E_L - E_M = C + F + E - (B - E) = C + (B + C) + E - (B - E) = 2C + 2E > 0,$$

  so $E_L > E_M$. Therefore, the midpoint approximation is more accurate than either the left or the right endpoint approximation.
- If $B < E$, then $E_M = E - B$. In this case,

  $$E_R - E_M = A + B + D - (E - B) = D + E + D - (E - B) = 2D + B > 0,$$

  so that $E_R > E_M$ while

  $$E_L - E_M = C + F + E - (E - B) = C + F + B > 0,$$

  so $E_L > E_M$. Therefore, the midpoint approximation is more accurate than either the right or the left endpoint approximation.
- If $B = E$, the midpoint approximation is exactly equal to the area.

Hence, for $B < E$, $B > E$, or $B = E$, the midpoint approximation is more accurate than either the left endpoint or the right endpoint approximation.

5.2 The Definite Integral

Preliminary Questions

1. What is $\int_3^5 dx$ [the function is $f(x) = 1$]?

SOLUTION $\int_3^5 dx = \int_3^5 1 \cdot dx = 1(5-3) = 2.$

2. Let $I = \int_2^7 f(x)\,dx$, where $f(x)$ is continuous. State whether true or false:

(a) I is the area between the graph and the x-axis over [2, 7].
(b) If $f(x) \ge 0$, then I is the area between the graph and the x-axis over [2, 7].
(c) If $f(x) \le 0$, then $-I$ is the area between the graph of $f(x)$ and the x-axis over [2, 7].

SOLUTION

(a) False. $\int_a^b f(x)\,dx$ is the *signed* area between the graph and the x-axis.
(b) True.
(c) True.

3. Explain graphically: $\int_0^\pi \cos x\,dx = 0$.

SOLUTION Because $\cos(\pi - x) = -\cos x$, the "negative" area between the graph of $y = \cos x$ and the x-axis over $[\frac{\pi}{2}, \pi]$ exactly cancels the "positive" area between the graph and the x-axis over $[0, \frac{\pi}{2}]$.

4. Which is negative, $\int_{-1}^{-5} 8\,dx$ or $\int_{-5}^{-1} 8\,dx$?

SOLUTION Because $-5 - (-1) = -4$, $\int_{-1}^{-5} 8\,dx$ is negative.

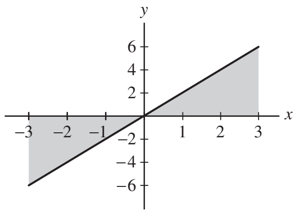
Exercises

In Exercises 1–10, draw a graph of the signed area represented by the integral and compute it using geometry.

1. $\int_{-3}^{3} 2x\,dx$

SOLUTION The region bounded by the graph of $y = 2x$ and the x-axis over the interval $[-3, 3]$ consists of two right triangles. One has area $\frac{1}{2}(3)(6) = 9$ below the axis, and the other has area $\frac{1}{2}(3)(6) = 9$ above the axis. Hence,

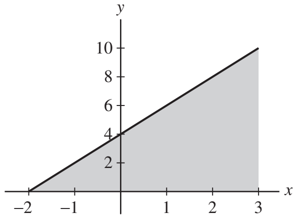
$$\int_{-3}^{3} 2x\,dx = 9 - 9 = 0.$$



2. $\int_{-2}^{3} (2x + 4)\,dx$

SOLUTION The region bounded by the graph of $y = 2x + 4$ and the x-axis over the interval $[-2, 3]$ consists of a single right triangle of area $\frac{1}{2}(5)(10) = 25$ above the axis. Hence,

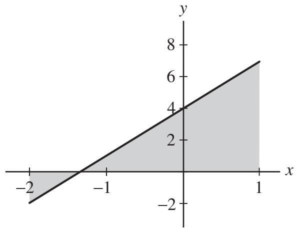
$$\int_{-2}^{3} (2x + 4)\,dx = 25.$$



3. $\int_{-2}^{1} (3x + 4)\, dx$

SOLUTION The region bounded by the graph of $y = 3x + 4$ and the x-axis over the interval $[-2, 1]$ consists of two right triangles. One has area $\frac{1}{2}(\frac{2}{3})(2) = \frac{2}{3}$ below the axis, and the other has area $\frac{1}{2}(\frac{7}{3})(7) = \frac{49}{6}$ above the axis. Hence,

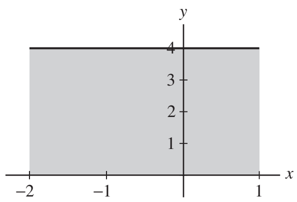
$$\int_{-2}^{1} (3x + 4)\, dx = \frac{49}{6} - \frac{2}{3} = \frac{15}{2}.$$



4. $\int_{-2}^{1} 4\, dx$

SOLUTION The region bounded by the graph of $y = 4$ and the x-axis over the interval $[-2, 1]$ is a rectangle of area $(3)(4) = 12$ above the axis. Hence,

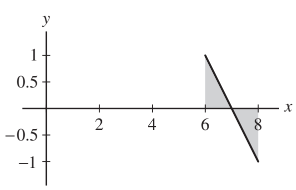
$$\int_{-2}^{1} 4\, dx = 12.$$



5. $\int_{6}^{8} (7 - x)\, dx$

SOLUTION The region bounded by the graph of $y = 7 - x$ and the x-axis over the interval $[6, 8]$ consists of two right triangles. One triangle has area $\frac{1}{2}(1)(1) = \frac{1}{2}$ above the axis, and the other has area $\frac{1}{2}(1)(1) = \frac{1}{2}$ below the axis. Hence,

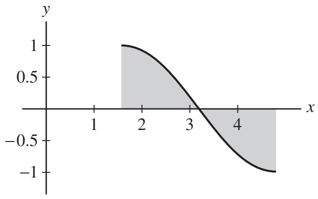
$$\int_{6}^{8} (7 - x)\, dx = \frac{1}{2} - \frac{1}{2} = 0.$$



6. $\int_{\pi/2}^{3\pi/2} \sin x\, dx$

SOLUTION The region bounded by the graph of $y = \sin x$ and the x-axis over the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$ consists of two parts of equal area, one above the axis and the other below the axis. Hence,

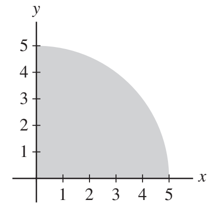
$$\int_{\pi/2}^{3\pi/2} \sin x\, dx = 0.$$



7. $\int_0^5 \sqrt{25 - x^2}\, dx$

SOLUTION The region bounded by the graph of $y = \sqrt{25 - x^2}$ and the x-axis over the interval $[0, 5]$ is one-quarter of a circle of radius 5. Hence,

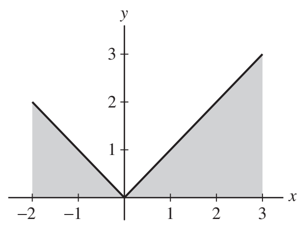
$$\int_0^5 \sqrt{25 - x^2}\, dx = \frac{1}{4}\pi(5)^2 = \frac{25\pi}{4}.$$



8. $\int_{-2}^3 |x|\, dx$

SOLUTION The region bounded by the graph of $y = |x|$ and the x-axis over the interval $[-2, 3]$ consists of two right triangles, both above the axis. One triangle has area $\frac{1}{2}(2)(2) = 2$, and the other has area $\frac{1}{2}(3)(3) = \frac{9}{2}$. Hence,

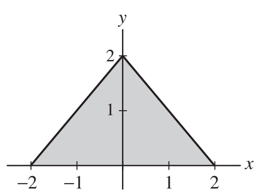
$$\int_{-2}^3 |x|\, dx = \frac{9}{2} + 2 = \frac{13}{2}.$$



9. $\int_{-2}^2 (2 - |x|)\, dx$

SOLUTION The region bounded by the graph of $y = 2 - |x|$ and the x-axis over the interval $[-2, 2]$ is a triangle above the axis with base 4 and height 2. Consequently,

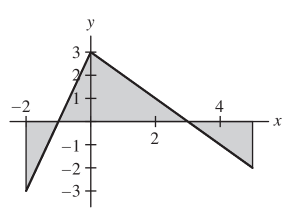
$$\int_{-2}^2 (2 - |x|)\, dx = \frac{1}{2}(2)(4) = 4.$$



10. $\int_{-2}^5 (3 + x - 2|x|)\, dx$

SOLUTION The region bounded by the graph of $y = 3 + x - 2|x|$ and the x-axis over the interval $[-2, 5]$ consists of a triangle below the axis with base 1 and height 3, a triangle above the axis of base 4 and height 3 and a triangle below the axis of base 2 and height 2. Consequently,

$$\int_{-2}^5 (3 + x - 2|x|)\, dx = -\frac{1}{2}(1)(3) + \frac{1}{2}(4)(3) - \frac{1}{2}(2)(2) = \frac{5}{2}.$$



11. Calculate $\int_0^{10} (8 - x)\, dx$ in two ways:

(a) As the limit $\lim_{N\to\infty} R_N$

(b) By sketching the relevant signed area and using geometry

SOLUTION Let $f(x) = 8 - x$ over [0, 10]. Consider the integral $\int_0^{10} f(x)\, dx = \int_0^{10} (8 - x)\, dx$.

(a) Let N be a positive integer and set $a = 0$, $b = 10$, $\Delta x = (b - a)/N = 10/N$. Also, let $x_k = a + k\Delta x = 10k/N$, $k = 1, 2, \ldots, N$ be the right endpoints of the N subintervals of [0, 10]. Then

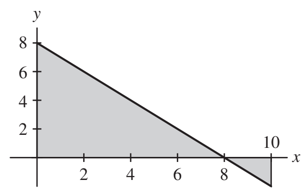
$$R_N = \Delta x \sum_{k=1}^{N} f(x_k) = \frac{10}{N} \sum_{k=1}^{N} \left(8 - \frac{10k}{N}\right) = \frac{10}{N}\left(8\left(\sum_{k=1}^{N} 1\right) - \frac{10}{N}\left(\sum_{k=1}^{N} k\right)\right)$$

$$= \frac{10}{N}\left(8N - \frac{10}{N}\left(\frac{N^2}{2} + \frac{N}{2}\right)\right) = 30 - \frac{50}{N}.$$

Hence $\lim_{N\to\infty} R_N = \lim_{N\to\infty} \left(30 - \frac{50}{N}\right) = 30.$

(b) The region bounded by the graph of $y = 8 - x$ and the x-axis over the interval [0, 10] consists of two right triangles. One triangle has area $\frac{1}{2}(8)(8) = 32$ above the axis, and the other has area $\frac{1}{2}(2)(2) = 2$ below the axis. Hence,

$$\int_0^{10} (8 - x)\, dx = 32 - 2 = 30.$$



12. Calculate $\int_{-1}^{4} (4x - 8)\, dx$ in two ways: As the limit $\lim_{N\to\infty} R_N$ and using geometry.

SOLUTION Let $f(x) = 4x - 8$ over [−1, 4]. Consider the integral $\int_{-1}^{4} f(x)\, dx = \int_{-1}^{4} (4x - 8)\, dx$.

- Let N be a positive integer and set $a = -1$, $b = 4$, $\Delta x = (b - a)/N = 5/N$. Then $x_k = a + k\Delta x = -1 + 5k/N$, $k = 1, 2, \ldots, N$ are the right endpoints of the N subintervals of [−1, 4]. Then

$$R_N = \Delta x \sum_{k=1}^{N} f(x_k) = \frac{5}{N} \sum_{k=1}^{N} \left(-4 + \frac{20k}{N} - 8\right) = -\frac{60}{N}\left(\sum_{k=1}^{N} 1\right) + \frac{100}{N^2}\left(\sum_{k=1}^{N} k\right)$$

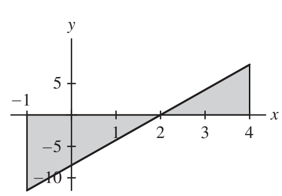
$$= -\frac{60}{N}(N) + \frac{100}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right)$$

$$= -60 + 50 + \frac{50}{N} = -10 + \frac{50}{N}.$$

  Hence $\lim_{N\to\infty} R_N = \lim_{N\to\infty} \left(-10 + \frac{50}{N}\right) = -10.$

- The region bounded by the graph of $y = 4x - 8$ and the x-axis over the interval [−1, 4] consists of a triangle below the axis with base 3 and height 12 and a triangle above the axis with base 2 and height 8. Hence,

$$\int_{-1}^{4} (4x - 8)\, dx = -\frac{1}{2}(3)(12) + \frac{1}{2}(2)(8) = -10.$$



In Exercises 13 and 14, refer to Figure 14.

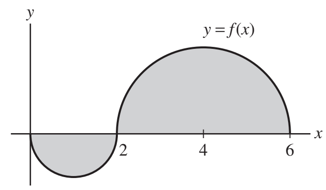


FIGURE 14 The two parts of the graph are semicircles.

13. Evaluate: (a) $\int_0^2 f(x)\,dx$ (b) $\int_0^6 f(x)\,dx$

SOLUTION Let $f(x)$ be given by Figure 14.

(a) The definite integral $\int_0^2 f(x)\,dx$ is the signed area of a semicircle of radius 1 which lies below the x-axis. Therefore,

$$\int_0^2 f(x)\,dx = -\frac{1}{2}\pi\,(1)^2 = -\frac{\pi}{2}.$$

(b) The definite integral $\int_0^6 f(x)\,dx$ is the signed area of a semicircle of radius 1 which lies below the x-axis and a semicircle of radius 2 which lies above the x-axis. Therefore,

$$\int_0^6 f(x)\,dx = \frac{1}{2}\pi\,(2)^2 - \frac{1}{2}\pi\,(1)^2 = \frac{3\pi}{2}.$$

14. Evaluate: (a) $\int_1^4 f(x)\,dx$ (b) $\int_1^6 |f(x)|\,dx$

SOLUTION Let $f(x)$ be given by Figure 14.

(a) The definite integral $\int_1^4 f(x)\,dx$ is the signed area of one-quarter of a circle of radius 1 which lies below the x-axis and one-quarter of a circle of radius 2 which lies above the x-axis. Therefore,

$$\int_1^4 f(x)\,dx = \frac{1}{4}\pi\,(2)^2 - \frac{1}{4}\pi\,(1)^2 = \frac{3}{4}\pi.$$

(b) The definite integral $\int_1^6 |f(x)|\,dx$ is the signed area of one-quarter of a circle of radius 1 and a semicircle of radius 2, both of which lie above the x-axis. Therefore,

$$\int_1^6 |f(x)|\;dx = \frac{1}{2}\pi\,(2)^2 + \frac{1}{4}\pi\,(1)^2 = \frac{9\pi}{4}.$$

In Exercises 15 and 16, refer to Figure 15.

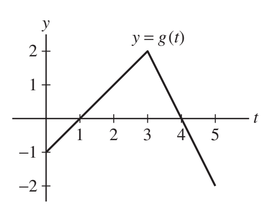


FIGURE 15

15. Evaluate $\int_0^3 g(t)\,dt$ and $\int_3^5 g(t)\,dt$.

SOLUTION

- The region bounded by the curve $y = g(x)$ and the x-axis over the interval [0, 3] is comprised of two right triangles, one with area $\frac{1}{2}$ below the axis, and one with area 2 above the axis. The definite integral is therefore equal to $2 - \frac{1}{2} = \frac{3}{2}$.
- The region bounded by the curve $y = g(x)$ and the x-axis over the interval [3, 5] is comprised of another two right triangles, one with area 1 above the axis and one with area 1 below the axis. The definite integral is therefore equal to 0.

16. Find a, b, and c such that $\int_0^a g(t)\,dt$ and $\int_b^c g(t)\,dt$ are as large as possible.

SOLUTION To make the value of $\int_0^a g(t)\,dt$ as large as possible, we want to include as much positive area as possible. This happens when we take $a = 4$. Now, to make the value of $\int_b^c g(t)\,dt$ as large as possible, we want to make sure to include all of the positive area and only the positive area. This happens when we take $b = 1$ and $c = 4$.

17. Describe the partition P and the set of sample points C for the Riemann sum shown in Figure 16. Compute the value of the Riemann sum.

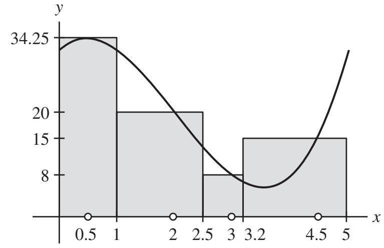


FIGURE 16

SOLUTION The partition P is defined by

$$x_0 = 0 \quad < \quad x_1 = 1 \quad < \quad x_2 = 2.5 \quad < \quad x_3 = 3.2 \quad < \quad x_4 = 5$$

The set of sample points is given by $C = \{c_1 = 0.5, c_2 = 2, c_3 = 3, c_4 = 4.5\}$. Finally, the value of the Riemann sum is

$$34.25(1-0) + 20(2.5-1) + 8(3.2-2.5) + 15(5-3.2) = 96.85.$$

18. Compute $R(f, P, C)$ for $f(x) = x^2 + x$ for the partition P and the set of sample points C in Figure 16.

SOLUTION

$$\begin{aligned} R(f, P, C) &= f(0.5)(1-0) + f(2)(2.5-1) + f(3)(3.2-2.5) + f(4.5)(5-3.2) \\ &= 34.25(1) + 20(1.5) + 8(0.7) + 15(1.8) = 96.85 \end{aligned}$$

In Exercises 19–22, calculate the Riemann sum $R(f, P, C)$ for the given function, partition, and choice of sample points. Also, sketch the graph of f and the rectangles corresponding to $R(f, P, C)$.

19. $f(x) = x, \quad P = \{1, 1.2, 1.5, 2\}, \quad C = \{1.1, 1.4, 1.9\}$

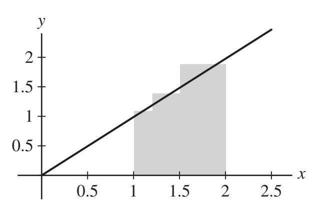
SOLUTION Let $f(x) = x$. With

$$P = \{x_0 = 1, x_1 = 1.2, x_2 = 1.5, x_3 = 2\} \qquad \text{and} \qquad C = \{c_1 = 1.1, c_2 = 1.4, c_3 = 1.9\},$$

we get

$$\begin{aligned} R(f, P, C) &= \Delta x_1 f(c_1) + \Delta x_2 f(c_2) + \Delta x_3 f(c_3) \\ &= (1.2-1)(1.1) + (1.5-1.2)(1.4) + (2-1.5)(1.9) = 1.59. \end{aligned}$$

Here is a sketch of the graph of f and the rectangles.



20. $f(x) = 2x + 3, \quad P = \{-4, -1, 1, 4, 8\}, \quad C = \{-3, 0, 2, 5\}$

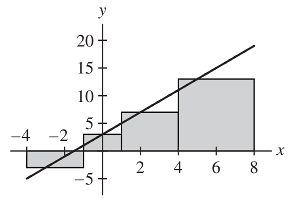
SOLUTION Let $f(x) = 2x + 3$. With

$$P = \{x_0 = -4, x_1 = -1, x_2 = 1, x_3 = 4, x_4 = 8\} \qquad \text{and} \qquad C = \{c_1 = -3, c_2 = 0, c_3 = 2, c_4 = 5\},$$

we get

$$\begin{aligned} R(f, P, C) &= \Delta x_1 f(c_1) + \Delta x_2 f(c_2) + \Delta x_3 f(c_3) + \Delta x_4 f(c_4) \\ &= (-1-(-4))(-3) + (1-(-1))(3) + (4-1)(7) + (8-4)(13) = 70. \end{aligned}$$

Here is a sketch of the graph of f and the rectangles.



21. $f(x) = x^2 + x, \quad P = \{2, 3, 4.5, 5\}, \quad C = \{2, 3.5, 5\}$

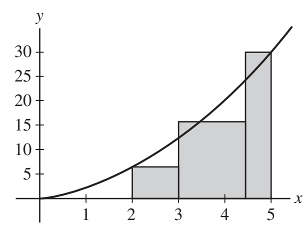
SOLUTION Let $f(x) = x^2 + x$. With

$$P = \{x_0 = 2, x_1 = 3, x_3 = 4.5, x_4 = 5\} \qquad \text{and} \qquad C = \{c_1 = 2, c_2 = 3.5, c_3 = 5\},$$

we get

$$\begin{aligned} R(f, P, C) &= \Delta x_1 f(c_1) + \Delta x_2 f(c_2) + \Delta x_3 f(c_3) \\ &= (3-2)(6) + (4.5-3)(15.75) + (5-4.5)(30) = 44.625. \end{aligned}$$

Here is a sketch of the graph of f and the rectangles.



22. $f(x) = \sin x, \quad P = \left\{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\right\}, \quad C = \{0.4, 0.7, 1.2\}$

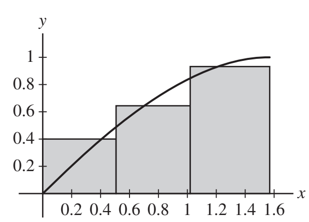
SOLUTION Let $f(x) = \sin x$. With

$$P = \left\{x_0 = 0, x_1 = \frac{\pi}{6}, x_3 = \frac{\pi}{3}, x_4 = \frac{\pi}{2}\right\} \qquad \text{and} \qquad C = \{c_1 = 0.4, c_2 = 0.7, c_3 = 1.2\},$$

we get

$$\begin{aligned} R(f, P, C) &= \Delta x_1 f(c_1) + \Delta x_2 f(c_2) + \Delta x_3 f(c_3) \\ &= \left(\frac{\pi}{6} - 0\right)(\sin 0.4) + \left(\frac{\pi}{3} - \frac{\pi}{6}\right)(\sin 0.7) + \left(\frac{\pi}{2} - \frac{\pi}{3}\right)(\sin 1.2) = 1.029225. \end{aligned}$$

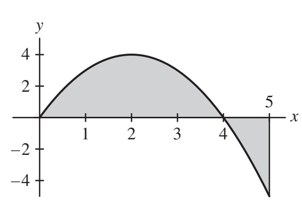
Here is a sketch of the graph of f and the rectangles.



In Exercises 23–28, sketch the signed area represented by the integral. Indicate the regions of positive and negative area.

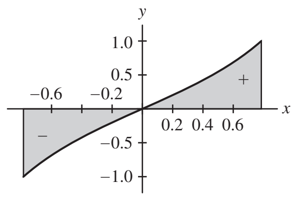
23. $\displaystyle\int_0^5 (4x - x^2)\, dx$

SOLUTION Here is a sketch of the signed area represented by the integral $\int_0^5 (4x - x^2)\, dx$.



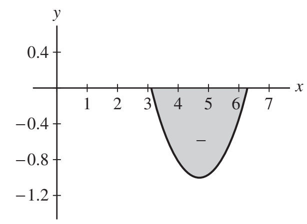
24. $\int_{-\pi/4}^{\pi/4} \tan x\,dx$

SOLUTION Here is a sketch of the signed area represented by the integral $\int_{-\pi/4}^{\pi/4} \tan x\,dx$.



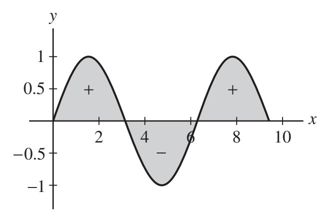
25. $\int_{\pi}^{2\pi} \sin x\,dx$

SOLUTION Here is a sketch of the signed area represented by the integral $\int_{\pi}^{2\pi} \sin x\,dx$.



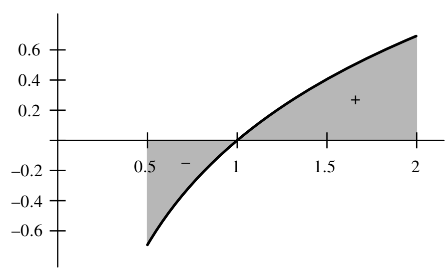
26. $\int_{0}^{3\pi} \sin x\,dx$

SOLUTION Here is a sketch of the signed area represented by the integral $\int_{0}^{3\pi} \sin x\,dx$.



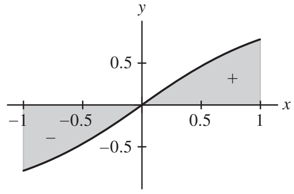
27. $\int_{1/2}^{2} \ln x\,dx$

SOLUTION Here is a sketch of the signed area represented by the integral $\int_{1/2}^{2} \ln x\,dx$.



28. $\int_{-1}^{1} \tan^{-1} x\,dx$

SOLUTION Here is a sketch of the signed area represented by the integral $\int_{-1}^{1} \tan^{-1} x\,dx$.



In Exercises 29–32, determine the sign of the integral without calculating it. Draw a graph if necessary.

29. $\int_{-2}^{1} x^4\,dx$

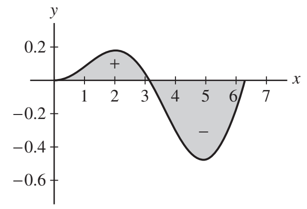
SOLUTION The integrand is always positive. The integral must therefore be positive, since the signed area has only positive part.

30. $\int_{-2}^{1} x^3\,dx$

SOLUTION By symmetry, the positive area from the interval $[0, 1]$ is cancelled by the negative area from $[-1, 0]$. With the interval $[-2, -1]$ contributing more negative area, the definite integral must be negative.

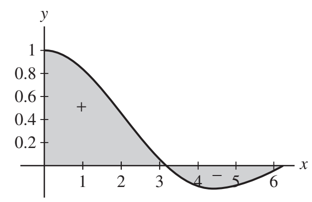
31. GU $\int_{0}^{2\pi} x\sin x\,dx$

SOLUTION As you can see from the graph below, the area below the axis is greater than the area above the axis. Thus, the definite integral is negative.



32. GU $\int_{0}^{2\pi} \frac{\sin x}{x}\,dx$

SOLUTION From the plot below, you can see that the area above the axis is bigger than the area below the axis, hence the integral is positive.



In Exercises 33–42, use properties of the integral and the formulas in the summary to calculate the integrals.

33. $\int_{0}^{4} (6t - 3)\,dt$

SOLUTION $\int_{0}^{4} (6t-3)\,dt = 6\int_{0}^{4} t\,dt - 3\int_{0}^{4} 1\,dt = 6\cdot\frac{1}{2}(4)^2 - 3(4-0) = 36.$

34. $\int_{-3}^{2} (4x + 7)\,dx$

SOLUTION

$$\begin{aligned}\int_{-3}^{2}(4x+7)\,dx &= 4\int_{-3}^{2} x\,dx + 7\int_{-3}^{2} dx\\ &= 4\left(\int_{-3}^{0} x\,dx + \int_{0}^{2} x\,dx\right) + 7(2-(-3))\\ &= 4\left(\int_{0}^{2} x\,dx - \int_{0}^{-3} x\,dx\right) + 35\\ &= 4\left(\frac{1}{2}2^2 - \frac{1}{2}(-3)^2\right) + 35 = 25.\end{aligned}$$

35. $\int_{0}^{9} x^2\,dx$

SOLUTION By formula (5), $\int_{0}^{9} x^2\,dx = \frac{1}{3}(9)^3 = 243.$

36. $\int_2^5 x^2\,dx$

SOLUTION $\int_2^5 x^2\,dx = \int_0^5 x^2\,dx - \int_0^2 x^2 dx = \frac{1}{3}(5)^3 - \frac{1}{3}(2)^3 = 39.$

37. $\int_0^1 (u^2 - 2u)\,du$

SOLUTION

$$\int_0^1 (u^2 - 2u)\,du = \int_0^1 u^2\,du - 2\int_0^1 u\,du = \frac{1}{3}(1)^3 - 2\left(\frac{1}{2}\right)(1)^2 = \frac{1}{3} - 1 = -\frac{2}{3}.$$

38. $\int_0^{1/2} (12y^2 + 6y)\,dy$

SOLUTION

$$\begin{aligned}\int_0^{1/2} (12y^2 + 6y)\,dy &= 12\int_0^{1/2} y^2\,dy + 6\int_0^{1/2} y\,dy \\ &= 12\cdot\frac{1}{3}\left(\frac{1}{2}\right)^3 + 6\cdot\frac{1}{2}\left(\frac{1}{2}\right)^2 \\ &= \frac{1}{2} + \frac{3}{4} = \frac{5}{4}.\end{aligned}$$

39. $\int_{-3}^1 (7t^2 + t + 1)\,dt$

SOLUTION First, write

$$\begin{aligned}\int_{-3}^1 (7t^2 + t + 1)\,dt &= \int_{-3}^0 (7t^2 + t + 1)\,dt + \int_0^1 (7t^2 + t + 1)\,dt \\ &= -\int_0^{-3} (7t^2 + t + 1)\,dt + \int_0^1 (7t^2 + t + 1)\,dt\end{aligned}$$

Then,

$$\begin{aligned}\int_{-3}^1 (7t^2 + t + 1)\,dt &= -\left(7\cdot\frac{1}{3}(-3)^3 + \frac{1}{2}(-3)^2 - 3\right) + \left(7\cdot\frac{1}{3}1^3 + \frac{1}{2}1^2 + 1\right) \\ &= -\left(-63 + \frac{9}{2} - 3\right) + \left(\frac{7}{3} + \frac{1}{2} + 1\right) = \frac{196}{3}.\end{aligned}$$

40. $\int_{-3}^3 (9x - 4x^2)\,dx$

SOLUTION First write

$$\begin{aligned}\int_{-3}^3 (9x - 4x^2)\,dx &= \int_{-3}^0 (9x - 4x^2)\,dx + \int_0^3 (9x - 4x^2)\,dx \\ &= -\int_0^{-3} (9x - 4x^2)\,dx + \int_0^3 (9x - 4x^2)\,dx.\end{aligned}$$

Then,

$$\begin{aligned}\int_{-3}^3 (9x - 4x^2)\,dx &= -\left(9\cdot\frac{1}{2}(-3)^2 - 4\cdot\frac{1}{3}(-3)^3\right) + \left(9\cdot\frac{1}{2}(3)^2 - 4\cdot\frac{1}{3}(3)^3\right) \\ &= -\left(\frac{81}{2} + 36\right) + \left(\frac{81}{2} - 36\right) = -72.\end{aligned}$$

41. $\int_{-a}^1 (x^2 + x)\,dx$

SOLUTION First, $\int_0^b (x^2 + x)\,dx = \int_0^b x^2\,dx + \int_0^b x\,dx = \frac{1}{3}b^3 + \frac{1}{2}b^2$. Therefore

$$\begin{aligned}\int_{-a}^1 (x^2 + x)\,dx &= \int_{-a}^0 (x^2 + x)\,dx + \int_0^1 (x^2 + x)\,dx = \int_0^1 (x^2 + x)\,dx - \int_0^{-a} (x^2 + x)\,dx \\ &= \left(\frac{1}{3}\cdot 1^3 + \frac{1}{2}\cdot 1^2\right) - \left(\frac{1}{3}(-a)^3 + \frac{1}{2}(-a)^2\right) = \frac{1}{3}a^3 - \frac{1}{2}a^2 + \frac{5}{6}.\end{aligned}$$

42. $\int_a^{a^2} x^2\,dx$

SOLUTION

$$\int_a^{a^2} x^2\,dx = \int_0^{a^2} x^2\,dx - \int_0^a x^2\,dx = \frac{1}{3}\left(a^2\right)^3 - \frac{1}{3}(a)^3 = \frac{1}{3}a^6 - \frac{1}{3}a^3.$$

In Exercises 43–47, calculate the integral, assuming that

$$\int_0^5 f(x)\,dx = 5, \qquad \int_0^5 g(x)\,dx = 12$$

43. $\int_0^5 (f(x) + g(x))\,dx$

SOLUTION $\int_0^5 (f(x) + g(x))\,dx = \int_0^5 f(x)\,dx + \int_0^5 g(x)\,dx = 5 + 12 = 17.$

44. $\int_0^5 \left(2f(x) - \frac{1}{3}g(x)\right) dx$

SOLUTION $\int_0^5 \left(2f(x) - \frac{1}{3}g(x)\right) dx = 2\int_0^5 f(x)\,dx - \frac{1}{3}\int_0^5 g(x)\,dx = 2(5) - \frac{1}{3}(12) = 6.$

45. $\int_5^0 g(x)\,dx$

SOLUTION $\int_5^0 g(x)\,dx = -\int_0^5 g(x)\,dx = -12.$

46. $\int_0^5 (f(x) - x)\,dx$

SOLUTION $\int_0^5 (f(x) - x)\,dx = \int_0^5 f(x)\,dx - \int_0^5 x\,dx = 5 - \frac{1}{2}(5)^2 = -\frac{15}{2}.$

47. Is it possible to calculate $\int_0^5 g(x)f(x)\,dx$ from the information given?

SOLUTION It is not possible to calculate $\int_0^5 g(x)f(x)\,dx$ from the information given.

48. Prove by computing the limit of right-endpoint approximations:

$$\boxed{\int_0^b x^3\,dx = \frac{b^4}{4}} \qquad \textbf{9}$$

SOLUTION Let $f(x) = x^3$, $a = 0$ and $\Delta x = (b - a)/N = b/N$. Then

$$R_N = \Delta x \sum_{k=1}^{N} f(x_k) = \frac{b}{N}\sum_{k=1}^{N}\left(k^3 \cdot \frac{b^3}{N^3}\right) = \frac{b^4}{N^4}\left(\sum_{k=1}^{N} k^3\right) = \frac{b^4}{N^4}\left(\frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{4}\right) = \frac{b^4}{4} + \frac{b^4}{2N} + \frac{b^4}{4N^2}.$$

Hence $\int_0^b x^3\,dx = \lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{b^4}{4} + \frac{b^4}{2N} + \frac{b^4}{4N^2}\right) = \frac{b^4}{4}.$

In Exercises 49–54, evaluate the integral using the formulas in the summary and Eq. (9).

49. $\int_0^3 x^3\,dx$

SOLUTION By Eq. (9), $\int_0^3 x^3\,dx = \frac{3^4}{4} = \frac{81}{4}.$

50. $\int_1^3 x^3\,dx$

SOLUTION $\int_1^3 x^3\,dx = \int_0^3 x^3\,dx - \int_0^1 x^3\,dx = \frac{1}{4}(3)^4 - \frac{1}{4}(1)^4 = 20.$

51. $\int_0^3 (x - x^3)\, dx$

SOLUTION $\int_0^3 (x - x^3)\, dx = \int_0^3 x\, dx - \int_0^3 x^3\, dx = \frac{1}{2}3^2 - \frac{1}{4}3^4 = -\frac{63}{4}.$

52. $\int_0^1 (2x^3 - x + 4)\, dx$

SOLUTION Applying the linearity of the definite integral, Eq. (9), the formula from Example 4 and the formula for the definite integral of a constant:

$$\int_0^1 (2x^3 - x + 4)\, dx = 2\int_0^1 x^3\, dx - \int_0^1 x\, dx + \int_0^1 4\, dx = 2 \cdot \frac{1}{4}(1)^4 - \frac{1}{2}(1)^2 + 4 = 4.$$

53. $\int_0^1 (12x^3 + 24x^2 - 8x)\, dx$

SOLUTION

$$\begin{aligned}\int_0^1 (12x^3 + 24x^2 - 8x)\, dx &= 12\int_0^1 x^3\, dx + 24\int_0^1 x^2 - 8\int_0^1 x\, dx \\ &= 12 \cdot \frac{1}{4}1^4 + 24 \cdot \frac{1}{3}1^3 - 8 \cdot \frac{1}{2}1^2 \\ &= 3 + 8 - 4 = 7\end{aligned}$$

54. $\int_{-2}^2 (2x^3 - 3x^2)\, dx$

SOLUTION

$$\begin{aligned}\int_{-2}^2 (2x^3 - 3x^2)\, dx &= \int_{-2}^0 (2x^3 - 3x^2)\, dx + \int_0^2 (2x^3 - 3x^2)\, dx \\ &= \int_0^2 (2x^3 - 3x^2)\, dx - \int_0^{-2} (2x^3 - 3x^2)\, dx \\ &= 2\int_0^2 x^3\, dx - 3\int_0^2 x^2\, dx - 2\int_0^{-2} x^3\, dx + 3\int_0^{-2} x^2\, dx \\ &= 2 \cdot \frac{1}{4}(2)^4 - 3 \cdot \frac{1}{3}(2)^3 - 2 \cdot \frac{1}{4}(-2)^4 + 3 \cdot \frac{1}{3}(-2)^3 \\ &= 8 - 8 - 8 - 8 = -16.\end{aligned}$$

In Exercises 55–58, calculate the integral, assuming that

$$\int_0^1 f(x)\, dx = 1, \qquad \int_0^2 f(x)\, dx = 4, \qquad \int_1^4 f(x)\, dx = 7$$

55. $\int_0^4 f(x)\, dx$

SOLUTION $\int_0^4 f(x)\, dx = \int_0^1 f(x)\, dx + \int_1^4 f(x)\, dx = 1 + 7 = 8.$

56. $\int_1^2 f(x)\, dx$

SOLUTION $\int_1^2 f(x)\, dx = \int_0^2 f(x)\, dx - \int_0^1 f(x)\, dx = 4 - 1 = 3.$

57. $\int_4^1 f(x)\, dx$

SOLUTION $\int_4^1 f(x)\, dx = -\int_1^4 f(x)\, dx = -7.$

58. $\int_2^4 f(x)\, dx$

SOLUTION From Exercise 55, $\int_0^4 f(x)\, dx = 8$. Accordingly,

$$\int_2^4 f(x)\, dx = \int_0^4 f(x)\, dx - \int_0^2 f(x)\, dx = 8 - 4 = 4.$$

In Exercises 59–62, express each integral as a single integral.

59. $\int_0^3 f(x)\,dx + \int_3^7 f(x)\,dx$

SOLUTION $\int_0^3 f(x)\,dx + \int_3^7 f(x)\,dx = \int_0^7 f(x)\,dx.$

60. $\int_2^9 f(x)\,dx - \int_4^9 f(x)\,dx$

SOLUTION $\int_2^9 f(x)\,dx - \int_4^9 f(x)\,dx = \left(\int_2^4 f(x)\,dx + \int_4^9 f(x)\,dx\right) - \int_4^9 f(x)\,dx = \int_2^4 f(x)\,dx.$

61. $\int_2^9 f(x)\,dx - \int_2^5 f(x)\,dx$

SOLUTION $\int_2^9 f(x)\,dx - \int_2^5 f(x)\,dx = \left(\int_2^5 f(x)\,dx + \int_5^9 f(x)\,dx\right) - \int_2^5 f(x)\,dx = \int_5^9 f(x)\,dx.$

62. $\int_7^3 f(x)\,dx + \int_3^9 f(x)\,dx$

SOLUTION $\int_7^3 f(x)\,dx + \int_3^9 f(x)\,dx = -\int_3^7 f(x)\,dx + \left(\int_3^7 f(x)\,dx + \int_7^9 f(x)\,dx\right) = \int_7^9 f(x)\,dx.$

In Exercises 63–66, calculate the integral, assuming that f is integrable and $\int_1^b f(x)\,dx = 1 - b^{-1}$ for all $b > 0$.

63. $\int_1^5 f(x)\,dx$

SOLUTION $\int_1^5 f(x)\,dx = 1 - 5^{-1} = \frac{4}{5}.$

64. $\int_3^5 f(x)\,dx$

SOLUTION $\int_3^5 f(x)\,dx = \int_1^5 f(x)\,dx - \int_1^3 f(x)\,dx = \left(1 - \frac{1}{5}\right) - \left(1 - \frac{1}{3}\right) = \frac{2}{15}.$

65. $\int_1^6 (3f(x) - 4)\,dx$

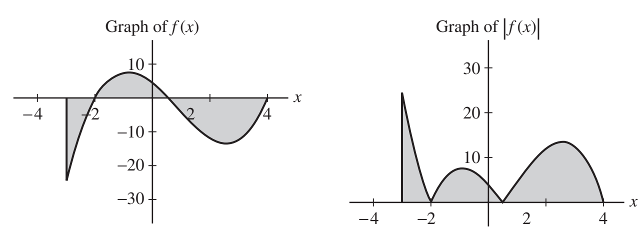
SOLUTION $\int_1^6 (3f(x) - 4)\,dx = 3\int_1^6 f(x)\,dx - 4\int_1^6 1\,dx = 3(1 - 6^{-1}) - 4(6 - 1) = -\frac{35}{2}.$

66. $\int_{1/2}^1 f(x)\,dx$

SOLUTION $\int_{1/2}^1 f(x)\,dx = -\int_1^{1/2} f(x)\,dx = -\left(1 - \left(\frac{1}{2}\right)^{-1}\right) = 1.$

67. Explain the difference in graphical interpretation between $\int_a^b f(x)\,dx$ and $\int_a^b |f(x)|\,dx$.

SOLUTION When $f(x)$ takes on both positive and negative values on $[a, b]$, $\int_a^b f(x)\,dx$ represents the signed area between $f(x)$ and the x-axis, whereas $\int_a^b |f(x)|\,dx$ represents the total (unsigned) area between $f(x)$ and the x-axis. Any negatively signed areas that were part of $\int_a^b f(x)\,dx$ are regarded as positive areas in $\int_a^b |f(x)|\,dx$. Here is a graphical example of this phenomenon.



68. Use the graphical interpretation of the definite integral to explain the inequality

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

where $f(x)$ is continuous. Explain also why equality holds if and only if either $f(x) \ge 0$ for all x or $f(x) \le 0$ for all x.

SOLUTION Let A_+ denote the unsigned area under the graph of $y = f(x)$ over the interval $[a, b]$ where $f(x) \ge 0$. Similarly, let A_- denote the unsigned area when $f(x) < 0$. Then

$$\int_a^b f(x)\,dx = A_+ - A_-.$$

Moreover,

$$\left|\int_a^b f(x)\,dx\right| \le A_+ + A_- = \int_a^b |f(x)|\,dx.$$

Equality holds if and only if one of the unsigned areas is equal to zero; in other words, if and only if either $f(x) \ge 0$ for all x or $f(x) \le 0$ for all x.

69. Let $f(x) = x$. Find an interval $[a, b]$ such that

$$\left|\int_a^b f(x)\,dx\right| = \frac{1}{2} \qquad \text{and} \qquad \int_a^b |f(x)|\,dx = \frac{3}{2}$$

SOLUTION If $a > 0$, then $f(x) \ge 0$ for all $x \in [a, b]$, so

$$\left|\int_a^b f(x)\,dx\right| = \int_a^b |f(x)|\,dx$$

by the previous exercise. We find a similar result if $b < 0$. Thus, we must have $a < 0$ and $b > 0$. Now,

$$\int_a^b |f(x)|\,dx = \frac{1}{2}a^2 + \frac{1}{2}b^2.$$

Because

$$\int_a^b f(x)\,dx = \frac{1}{2}b^2 - \frac{1}{2}a^2,$$

then

$$\left|\int_a^b f(x)\,dx\right| = \frac{1}{2}|b^2 - a^2|.$$

If $b^2 > a^2$, then

$$\frac{1}{2}a^2 + \frac{1}{2}b^2 = \frac{3}{2} \quad \text{and} \quad \frac{1}{2}(b^2 - a^2) = \frac{1}{2}$$

yield $a = -1$ and $b = \sqrt{2}$. On the other hand, if $b^2 < a^2$, then

$$\frac{1}{2}a^2 + \frac{1}{2}b^2 = \frac{3}{2} \quad \text{and} \quad \frac{1}{2}(a^2 - b^2) = \frac{1}{2}$$

yield $a = -\sqrt{2}$ and $b = 1$.

70. Evaluate $I = \int_0^{2\pi} \sin^2 x\,dx$ and $J = \int_0^{2\pi} \cos^2 x\,dx$ as follows. First show with a graph that $I = J$. Then prove that $I + J = 2\pi$.

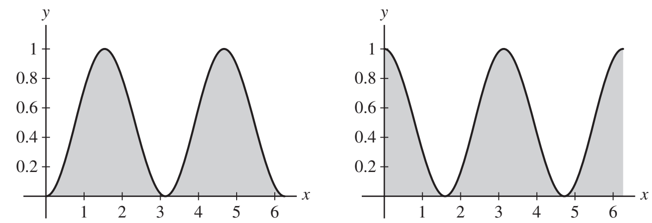
SOLUTION The graphs of $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$ are shown below at the left and right, respectively. It is clear that the shaded areas in the two graphs are equal, thus

$$I = \int_0^{2\pi} \sin^2 x\,dx = \int_0^{2\pi} \cos^2 x\,dx = J.$$

Now, using the fundamental trigonometric identity, we find

$$I + J = \int_0^{2\pi} (\sin^2 x + \cos^2 x)\,dx = \int_0^{2\pi} 1 \cdot dx = 2\pi.$$

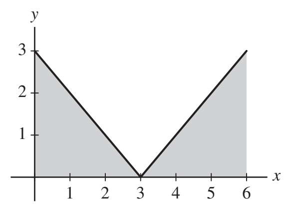
Combining this last result with $I = J$ yields $I = J = \pi$.



In Exercises 71–74, calculate the integral.

71. $\int_0^6 |3 - x|\,dx$

SOLUTION Over the interval, the region between the curve and the interval [0, 6] consists of two triangles above the x axis, each of which has height 3 and width 3, and so area $\frac{9}{2}$. The total area, hence the definite integral, is 9.

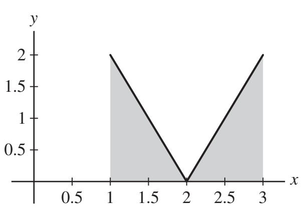


Alternately,

$$\int_0^6 |3 - x|\,dx = \int_0^3 (3 - x)\,dx + \int_3^6 (x - 3)\,dx$$
$$= 3\int_0^3 dx - \int_0^3 x\,dx + \left(\int_0^6 x\,dx - \int_0^3 x\,dx\right) - 3\int_3^6 dx$$
$$= 9 - \frac{1}{2}3^2 + \frac{1}{2}6^2 - \frac{1}{2}3^2 - 9 = 9.$$

72. $\int_1^3 |2x - 4|\,dx$

SOLUTION The area between $|2x - 4|$ and the x axis consists of two triangles above the x-axis, each with width 1 and height 2, and hence with area 1. The total area, and hence the definite integral, is 2.



Alternately,

$$\int_1^3 |2x - 4|\,dx = \int_1^2 (4 - 2x)\,dx + \int_2^3 (2x - 4)\,dx$$
$$= 4\int_1^2 dx - 2\left(\int_0^2 x\,dx - \int_0^1 x\,dx\right) + 2\left(\int_0^3 x\,dx - \int_0^2 x\,dx\right) - 4\int_2^3 dx$$
$$= 4 - 2\left(\frac{1}{2}2^2 - \frac{1}{2}1^2\right) + 2\left(\frac{1}{2}3^2 - \frac{1}{2}2^2\right) - 4 = 2.$$

73. $\int_{-1}^1 |x^3|\,dx$

SOLUTION

$$|x^3| = \begin{cases} x^3 & x \geq 0 \\ -x^3 & x < 0. \end{cases}$$

Therefore,

$$\int_{-1}^{1} |x^3|\,dx = \int_{-1}^{0} -x^3\,dx + \int_{0}^{1} x^3\,dx = \int_{0}^{-1} x^3\,dx + \int_{0}^{1} x^3\,dx = \frac{1}{4}(-1)^4 + \frac{1}{4}(1)^4 = \frac{1}{2}.$$

74. $\int_0^2 |x^2 - 1|\,dx$

SOLUTION

$$|x^2 - 1| = \begin{cases} x^2 - 1 & 1 \le x \le 2 \\ -(x^2 - 1) & 0 \le x < 1. \end{cases}$$

Therefore,

$$\begin{aligned} \int_0^2 |x^2 - 1|\,dx &= \int_0^1 (1 - x^2)\,dx + \int_1^2 (x^2 - 1)\,dx \\ &= \int_0^1 dx - \int_0^1 x^2\,dx + \left(\int_0^2 x^2\,dx - \int_0^1 x^2\,dx\right) - \int_1^2 1\,dx \\ &= 1 - \frac{1}{3}(1) + \left(\frac{1}{3}(8) - \frac{1}{3}(1)\right) - 1 = 2. \end{aligned}$$

75. Use the Comparison Theorem to show that

$$\int_0^1 x^5\,dx \le \int_0^1 x^4\,dx, \qquad \int_1^2 x^4\,dx \le \int_1^2 x^5\,dx$$

SOLUTION On the interval [0, 1], $x^5 \le x^4$, so, by Theorem 5,

$$\int_0^1 x^5\,dx \le \int_0^1 x^4\,dx.$$

On the other hand, $x^4 \le x^5$ for $x \in [1, 2]$, so, by the same Theorem,

$$\int_1^2 x^4\,dx \le \int_1^2 x^5\,dx.$$

76. Prove that $\frac{1}{3} \le \int_4^6 \frac{1}{x}\,dx \le \frac{1}{2}$.

SOLUTION On the interval [4, 6], $\frac{1}{6} \le \frac{1}{x}$, so, by Theorem 5,

$$\frac{1}{3} = \int_4^6 \frac{1}{6}\,dx \le \int_4^6 \frac{1}{x}\,dx.$$

On the other hand, $\frac{1}{x} \le \frac{1}{4}$ on the interval [4, 6], so

$$\int_4^6 \frac{1}{x}\,dx \le \int_4^6 \frac{1}{4}\,dx = \frac{1}{4}(6 - 4) = \frac{1}{2}.$$

Therefore $\frac{1}{3} \le \int_4^6 \frac{1}{x}\,dx \le \frac{1}{2}$, as desired.

77. Prove that $0.0198 \le \int_{0.2}^{0.3} \sin x\,dx \le 0.0296$. *Hint:* Show that $0.198 \le \sin x \le 0.296$ for x in [0.2, 0.3].

SOLUTION For $0 \le x \le \frac{\pi}{6} \approx 0.52$, we have $\frac{d}{dx}(\sin x) = \cos x > 0$. Hence $\sin x$ is increasing on [0.2, 0.3]. Accordingly, for $0.2 \le x \le 0.3$, we have

$$m = 0.198 \le 0.19867 \approx \sin 0.2 \le \sin x \le \sin 0.3 \approx 0.29552 \le 0.296 = M$$

Therefore, by the Comparison Theorem, we have

$$0.0198 = m(0.3 - 0.2) = \int_{0.2}^{0.3} m\,dx \le \int_{0.2}^{0.3} \sin x\,dx \le \int_{0.2}^{0.3} M\,dx = M(0.3 - 0.2) = 0.0296.$$

78. Prove that $0.277 \le \int_{\pi/8}^{\pi/4} \cos x\, dx \le 0.363$.

SOLUTION $\cos x$ is decreasing on the interval $[\pi/8, \pi/4]$. Hence, for $\pi/8 \le x \le \pi/4$,

$$\cos(\pi/4) \le \cos x \le \cos(\pi/8).$$

Since $\cos(\pi/4) = \sqrt{2}/2$,

$$0.277 \le \frac{\pi}{8} \cdot \frac{\sqrt{2}}{2} = \int_{\pi/8}^{\pi/4} \frac{\sqrt{2}}{2}\, dx \le \int_{\pi/8}^{\pi/4} \cos x\, dx.$$

Since $\cos(\pi/8) \le 0.924$,

$$\int_{\pi/8}^{\pi/4} \cos x\, dx \le \int_{\pi/8}^{\pi/4} 0.924\, dx = \frac{\pi}{8}(0.924) \le 0.363.$$

Therefore $0.277 \le \int_{\pi/8}^{\pi/4} \cos x \le 0.363$.

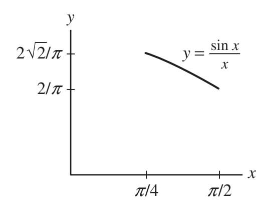
79. Prove that $0 \le \int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\, dx \le \frac{\sqrt{2}}{2}$.

SOLUTION Let

$$f(x) = \frac{\sin x}{x}.$$

As we can see in the sketch below, $f(x)$ is decreasing on the interval $[\pi/4, \pi/2]$. Therefore $f(x) \le f(\pi/4)$ for all x in $[\pi/4, \pi/2]$. $f(\pi/4) = \frac{2\sqrt{2}}{\pi}$, so:

$$\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\, dx \le \int_{\pi/4}^{\pi/2} \frac{2\sqrt{2}}{\pi}\, dx = \frac{\pi}{4}\frac{2\sqrt{2}}{\pi} = \frac{\sqrt{2}}{2}.$$



80. Find upper and lower bounds for $\int_0^1 \frac{dx}{\sqrt{5x^3+4}}$.

SOLUTION Let

$$f(x) = \frac{1}{\sqrt{5x^3+4}}.$$

$f(x)$ is decreasing for x on the interval $[0, 1]$, so $f(1) \le f(x) \le f(0)$ for all x in $[0, 1]$. $f(0) = \frac{1}{2}$ and $f(1) = \frac{1}{3}$, so

$$\int_0^1 \frac{1}{3}\, dx \le \int_0^1 f(x)\, dx \le \int_0^1 \frac{1}{2}\, dx$$
$$\frac{1}{3} \le \int_0^1 f(x)\, dx \le \frac{1}{2}.$$

81. Suppose that $f(x) \le g(x)$ on $[a, b]$. By the Comparison Theorem, $\int_a^b f(x)\, dx \le \int_a^b g(x)\, dx$. Is it also true that $f'(x) \le g'(x)$ for $x \in [a, b]$? If not, give a counterexample.

SOLUTION The assertion $f'(x) \le g'(x)$ is false. Consider $a = 0$, $b = 1$, $f(x) = x$, $g(x) = 2$. $f(x) \le g(x)$ for all x in the interval $[0, 1]$, but $f'(x) = 1$ while $g'(x) = 0$ for all x.

82. State whether true or false. If false, sketch the graph of a counterexample.

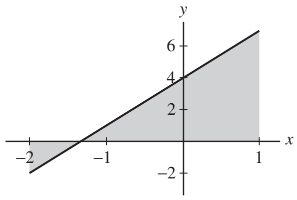
(a) If $f(x) > 0$, then $\int_a^b f(x)\, dx > 0$.

(b) If $\int_a^b f(x)\, dx > 0$, then $f(x) > 0$.

SOLUTION

(a) It is true that if $f(x) > 0$ for $x \in [a, b]$, then $\int_a^b f(x)\,dx > 0$.

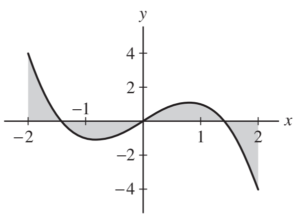
(b) It is *false* that if $\int_a^b f(x)\,dx > 0$, then $f(x) > 0$ for $x \in [a, b]$. Indeed, in Exercise 3, we saw that $\int_{-2}^{1}(3x + 4)\,dx = 7.5 > 0$, yet $f(-2) = -2 < 0$. Here is the graph from that exercise.



Further Insights and Challenges

83. Explain graphically: If $f(x)$ is an odd function, then
$\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$

SOLUTION If f is an odd function, then $f(-x) = -f(x)$ for all x. Accordingly, for every positively signed area in the right half-plane where f is above the x-axis, there is a corresponding negatively signed area in the left half-plane where f is below the x-axis. Similarly, for every negatively signed area in the right half-plane where f is below the x-axis, there is a corresponding positively signed area in the left half-plane where f is above the x-axis. We conclude that the net area between the graph of f and the x-axis over $[-a, a]$ is 0, since the positively signed areas and negatively signed areas cancel each other out exactly.



84. Compute $\displaystyle\int_{-1}^{1} \sin(\sin(x))(\sin^2(x) + 1)\,dx$.

SOLUTION Let $f(x) = \sin(\sin(x))(\sin^2(x) + 1))$. $\sin x$ is an odd function, while $\sin^2 x$ is an even function, so:

$$\begin{aligned} f(-x) &= \sin(\sin(-x))(\sin^2(-x) + 1) = \sin(-\sin(x))(\sin^2(x) + 1) \\ &= -\sin(\sin(x))(\sin^2(x) + 1) = -f(x). \end{aligned}$$

Therefore, $f(x)$ is an odd function. The function is odd and the interval is symmetric around the origin so, by the previous exercise, the integral must be zero.

85. Let k and b be positive. Show, by comparing the right-endpoint approximations, that

$$\int_0^b x^k\,dx = b^{k+1}\int_0^1 x^k\,dx$$

SOLUTION Let k and b be any positive numbers. Let $f(x) = x^k$ on $[0, b]$. Since f is continuous, both $\int_0^b f(x)\,dx$ and $\int_0^1 f(x)\,dx$ exist. Let N be a positive integer and set $\Delta x = (b - 0)/N = b/N$. Let $x_j = a + j\Delta x = bj/N$, $j = 1, 2, \ldots, N$ be the right endpoints of the N subintervals of $[0, b]$. Then the right-endpoint approximation to $\int_0^b f(x)\,dx = \int_0^b x^k\,dx$ is

$$R_N = \Delta x \sum_{j=1}^{N} f(x_j) = \frac{b}{N}\sum_{j=1}^{N}\left(\frac{bj}{N}\right)^k = b^{k+1}\left(\frac{1}{N^{k+1}}\sum_{j=1}^{N} j^k\right).$$

In particular, if $b = 1$ above, then the right-endpoint approximation to $\int_0^1 f(x)\,dx = \int_0^1 x^k\,dx$ is

$$S_N = \Delta x \sum_{j=1}^{N} f(x_j) = \frac{1}{N}\sum_{j=1}^{N}\left(\frac{j}{N}\right)^k = \frac{1}{N^{k+1}}\sum_{j=1}^{N} j^k = \frac{1}{b^{k+1}}R_N$$

In other words, $R_N = b^{k+1} S_N$. Therefore,

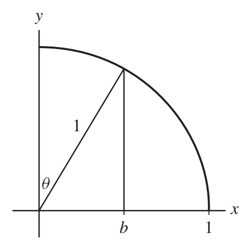
$$\int_0^b x^k\,dx = \lim_{N\to\infty} R_N = \lim_{N\to\infty} b^{k+1} S_N = b^{k+1} \lim_{N\to\infty} S_N = b^{k+1} \int_0^1 x^k\,dx.$$

86. Verify for $0 \le b \le 1$ by interpreting in terms of area:

$$\int_0^b \sqrt{1-x^2}\,dx = \frac{1}{2} b\sqrt{1-b^2} + \frac{1}{2}\sin^{-1} b$$

SOLUTION The function $f(x) = \sqrt{1-x^2}$ is the quarter circle of radius 1 in the first quadrant. For $0 \le b \le 1$, the area represented by the integral $\int_0^b \sqrt{1-x^2}\,dx$ can be divided into two parts. The area of the triangular part is $\frac{1}{2}(b)\sqrt{1-b^2}$ using the Pythagorean Theorem. The area of the sector with angle θ where $\sin\theta = b$, is given by $\frac{1}{2}(1)^2(\theta)$. Thus

$$\int_0^b \sqrt{1-x^2}\,dx = \frac{1}{2} b\sqrt{1-b^2} + \frac{1}{2}\theta = \frac{1}{2} b\sqrt{1-b^2} + \frac{1}{2}\sin^{-1} b.$$



87. Suppose that f and g are continuous functions such that, *for all* a,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} g(x)\,dx$$

Give an *intuitive* argument showing that $f(0) = g(0)$. Explain your idea with a graph.

SOLUTION Let $c = -b$. Since $b < 0$, $c > 0$, so by Eq. (5),

$$\int_0^c x^2\,dx = \frac{1}{3}c^3.$$

Furthermore, x^2 is an even function, so symmetry of the areas gives

$$\int_{-c}^0 x^2\,dx = \int_0^c x^2\,dx.$$

Finally,

$$\int_0^b x^2\,dx = \int_0^{-c} x^2\,dx = -\int_{-c}^0 x^2\,dx = -\int_0^c x^2\,dx = -\frac{1}{3}c^3 = \frac{1}{3}b^3.$$

88. Theorem 4 remains true without the assumption $a \le b \le c$. Verify this for the cases $b < a < c$ and $c < a < b$.

SOLUTION The additivity property of definite integrals states for $a \le b \le c$, we have $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$.

- Suppose that we have $b < a < c$. By the additivity property, we have $\int_b^c f(x)\,dx = \int_b^a f(x)\,dx + \int_a^c f(x)\,dx$. Therefore, $\int_a^c f(x)\,dx = \int_b^c f(x)\,dx - \int_b^a f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$.
- Now suppose that we have $c < a < b$. By the additivity property, we have $\int_c^b f(x)\,dx = \int_c^a f(x)\,dx + \int_a^b f(x)\,dx$. Therefore, $\int_a^c f(x)\,dx = -\int_c^a f(x)\,dx = \int_a^b f(x)\,dx - \int_c^b f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$.
- Hence the additivity property holds for all real numbers a, b, and c, regardless of their relationship amongst each other.

5.3 The Fundamental Theorem of Calculus, Part I

Preliminary Questions

1. Suppose that $F'(x) = f(x)$ and $F(0) = 3$, $F(2) = 7$.

(a) What is the area under $y = f(x)$ over $[0, 2]$ if $f(x) \geq 0$?

(b) What is the graphical interpretation of $F(2) - F(0)$ if $f(x)$ takes on both positive and negative values?

SOLUTION

(a) If $f(x) \geq 0$ over $[0, 2]$, then the area under $y = f(x)$ is $F(2) - F(0) = 7 - 3 = 4$.

(b) If $f(x)$ takes on both positive and negative values, then $F(2) - F(0)$ gives the signed area between $y = f(x)$ and the x-axis.

2. Suppose that $f(x)$ is a *negative* function with antiderivative F such that $F(1) = 7$ and $F(3) = 4$. What is the area (a positive number) between the x-axis and the graph of $f(x)$ over $[1, 3]$?

SOLUTION $\int_1^3 f(x)\,dx$ represents the *signed* area bounded by the curve and the interval $[1, 3]$. Since $f(x)$ is negative on $[1, 3]$, $\int_1^3 f(x)\,dx$ is the negative of the area. Therefore, if A is the area between the x-axis and the graph of $f(x)$, we have:

$$A = -\int_1^3 f(x)\,dx = -\left(F(3) - F(1)\right) = -(4 - 7) = -(-3) = 3.$$

3. Are the following statements true or false? Explain.

(a) FTC I is valid only for positive functions.

(b) To use FTC I, you have to choose the right antiderivative.

(c) If you cannot find an antiderivative of $f(x)$, then the definite integral does not exist.

SOLUTION

(a) False. The FTC I is valid for continuous functions.

(b) False. The FTC I works for any antiderivative of the integrand.

(c) False. If you cannot find an antiderivative of the integrand, you cannot use the FTC I to evaluate the definite integral, but the definite integral may still exist.

4. Evaluate $\int_2^9 f'(x)\,dx$ where $f(x)$ is differentiable and $f(2) = f(9) = 4$.

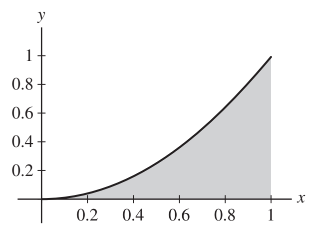
SOLUTION Because f is differentiable, $\int_2^9 f'(x)\,dx = f(9) - f(2) = 4 - 4 = 0$.

Exercises

In Exercises 1–4, sketch the region under the graph of the function and find its area using FTC I.

1. $f(x) = x^2$, $[0, 1]$

SOLUTION

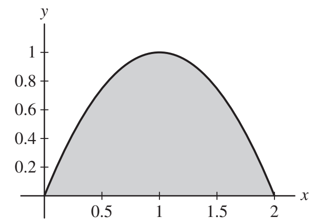


We have the area

$$A = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}.$$

2. $f(x) = 2x - x^2, \quad [0, 2]$

SOLUTION

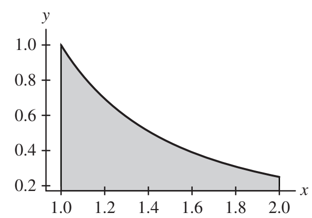


Let A be the area indicated. Then:

$$A = \int_0^2 (2x - x^2)\,dx = \int_0^2 2x\,dx - \int_0^2 x^2\,dx = x^2\Big|_0^2 - \frac{1}{3}x^3\Big|_0^2 = (4-0) - \left(\frac{8}{3} - 0\right) = \frac{4}{3}.$$

3. $f(x) = x^{-2}, \quad [1, 2]$

SOLUTION

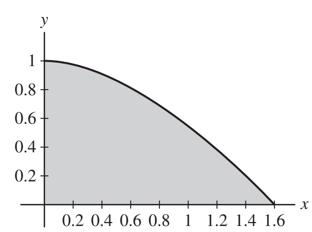


We have the area

$$A = \int_1^2 x^{-2}\,dx = \frac{x^{-1}}{-1}\Big|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}.$$

4. $f(x) = \cos x, \quad \left[0, \frac{\pi}{2}\right]$

SOLUTION



Let A be the shaded area. Then

$$A = \int_0^{\pi/2} \cos x\,dx = \sin x\Big|_0^{\pi/2} = 1 - 0 = 1.$$

In Exercises 5–42, evaluate the integral using FTC I.

5. $\displaystyle\int_3^6 x\,dx$

SOLUTION $\displaystyle\int_3^6 x\,dx = \frac{1}{2}x^2\Big|_3^6 = \frac{1}{2}(6)^2 - \frac{1}{2}(3)^2 = \frac{27}{2}.$

6. $\displaystyle\int_0^9 2\,dx$

SOLUTION $\displaystyle\int_0^9 2\,dx = 2x\Big|_0^9 = 2(9) - 2(0) = 18.$

7. $\displaystyle\int_0^1 (4x - 9x^2)\,dx$

SOLUTION $\displaystyle\int_0^1 (4x - 9x^2)\,dx = (2x^2 - 3x^3)\Big|_0^1 = (2-3) - (0-0) = -1.$

8. $\int_{-3}^{2} u^2\, du$

SOLUTION $\int_{-3}^{2} u^2\, du = \frac{1}{3}u^3\Big|_{-3}^{2} = \frac{1}{3}(2)^3 - \frac{1}{3}(-3)^3 = \frac{35}{3}.$

9. $\int_{0}^{2} (12x^5 + 3x^2 - 4x)\, dx$

SOLUTION $\int_{0}^{2} (12x^5 + 3x^2 - 4x)\, dx = (2x^6 + x^3 - 2x^2)\Big|_{0}^{2} = (128 + 8 - 8) - (0 + 0 - 0) = 128.$

10. $\int_{-2}^{2} (10x^9 + 3x^5)\, dx$

SOLUTION $\int_{-2}^{2} (10x^9 + 3x^5)\, dx = \left(x^{10} + \frac{1}{2}x^6\right)\Big|_{-2}^{2} = \left(2^{10} + \frac{1}{2}2^6\right) - \left(2^{10} + \frac{1}{2}2^6\right) = 0.$

11. $\int_{3}^{0} (2t^3 - 6t^2)\, dt$

SOLUTION $\int_{3}^{0} (2t^3 - 6t^2)\, dt = \left(\frac{1}{2}t^4 - 2t^3\right)\Big|_{3}^{0} = (0 - 0) - \left(\frac{81}{2} - 54\right) = \frac{27}{2}.$

12. $\int_{-1}^{1} (5u^4 + u^2 - u)\, du$

SOLUTION $\int_{-1}^{1} (5u^4 + u^2 - u)\, du = \left(u^5 + \frac{1}{3}u^3 - \frac{1}{2}u^2\right)\Big|_{-1}^{1} = \left(1 + \frac{1}{3} - \frac{1}{2}\right) - \left(-1 - \frac{1}{3} - \frac{1}{2}\right) = \frac{8}{3}.$

13. $\int_{0}^{4} \sqrt{y}\, dy$

SOLUTION $\int_{0}^{4} \sqrt{y}\, dy = \int_{0}^{4} y^{1/2}\, dy = \frac{2}{3}y^{3/2}\Big|_{0}^{4} = \frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{16}{3}.$

14. $\int_{1}^{8} x^{4/3}\, dx$

SOLUTION $\int_{1}^{8} x^{4/3}\, dx = \frac{3}{7}x^{7/3}\Big|_{1}^{8} = \frac{3}{7}(128 - 1) = \frac{381}{7}.$

15. $\int_{1/16}^{1} t^{1/4}\, dt$

SOLUTION $\int_{1/16}^{1} t^{1/4}\, dt = \frac{4}{5}t^{5/4}\Big|_{1/16}^{1} = \frac{4}{5} - \frac{1}{40} = \frac{31}{40}.$

16. $\int_{4}^{1} t^{5/2}\, dt$

SOLUTION $\int_{4}^{1} t^{5/2}\, dt = \frac{2}{7}t^{7/2}\Big|_{4}^{1} = \frac{2}{7}(1 - 128) = -\frac{254}{7}.$

17. $\int_{1}^{3} \frac{dt}{t^2}$

SOLUTION $\int_{1}^{3} \frac{dt}{t^2} = \int_{1}^{3} t^{-2}\, dt = -t^{-1}\Big|_{1}^{3} = -\frac{1}{3} + 1 = \frac{2}{3}.$

18. $\int_{1}^{4} x^{-4}\, dx$

SOLUTION $\int_{1}^{4} x^{-4}\, dx = -\frac{1}{3}x^{-3}\Big|_{1}^{4} = -\frac{1}{3}(4)^{-3} + \frac{1}{3} = \frac{21}{64}.$

19. $\int_{1/2}^{1} \frac{8}{x^3}\, dx$

SOLUTION $\int_{1/2}^{1} \frac{8}{x^3}\, dx = \int_{1/2}^{1} 8x^{-3}\, dx = -4x^{-2}\Big|_{1/2}^{1} = -4 + 16 = 12.$

20. $\int_{-2}^{-1} \frac{1}{x^3}\,dx$

SOLUTION $\int_{-2}^{-1} \frac{1}{x^3}\,dx = -\frac{1}{2}x^{-2}\Big|_{-2}^{-1} = -\frac{1}{2}(-1)^{-2} + \frac{1}{2}(-2)^{-2} = -\frac{3}{8}.$

21. $\int_{1}^{2} (x^2 - x^{-2})\,dx$

SOLUTION $\int_{1}^{2} (x^2 - x^{-2})\,dx = \left(\frac{1}{3}x^3 + x^{-1}\right)\Big|_1^2 = \left(\frac{8}{3} + \frac{1}{2}\right) - \left(\frac{1}{3} + 1\right) = \frac{11}{6}.$

22. $\int_{1}^{9} t^{-1/2}\,dt$

SOLUTION $\int_{1}^{9} t^{-1/2}\,dt = 2t^{1/2}\Big|_1^9 = 2(9)^{1/2} - 2(1)^{1/2} = 4.$

23. $\int_{1}^{27} \frac{t+1}{\sqrt{t}}\,dt$

SOLUTION

$$\int_{1}^{27} \frac{t+1}{\sqrt{t}}\,dt = \int_{1}^{27} (t^{1/2} + t^{-1/2})\,dt = \left(\frac{2}{3}t^{3/2} + 2t^{1/2}\right)\Big|_1^{27}$$
$$= \left(\frac{2}{3}(81\sqrt{3}) + 6\sqrt{3}\right) - \left(\frac{2}{3} + 2\right) = 60\sqrt{3} - \frac{8}{3}.$$

24. $\int_{8/27}^{1} \frac{10t^{4/3} - 8t^{1/3}}{t^2}\,dt$

SOLUTION

$$\int_{8/27}^{1} \frac{10t^{4/3} - 8t^{1/3}}{t^2}\,dt = \int_{8/27}^{1} (10t^{-2/3} - 8t^{-5/3})\,dt$$
$$= (30t^{1/3} + 12t^{-2/3})\Big|_{8/27}^{1} = (30 + 12) - (20 + 27) = -5.$$

25. $\int_{\pi/4}^{3\pi/4} \sin\theta\,d\theta$

SOLUTION $\int_{\pi/4}^{3\pi/4} \sin\theta\,d\theta = -\cos\theta\Big|_{\pi/4}^{3\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}.$

26. $\int_{2\pi}^{4\pi} \sin x\,dx$

SOLUTION $\int_{2\pi}^{4\pi} \sin x\,dx = -\cos x\Big|_{2\pi}^{4\pi} = -1 - (-1) = 0.$

27. $\int_{0}^{\pi/2} \cos\left(\frac{1}{3}\theta\right)d\theta$

SOLUTION $\int_{0}^{\pi/2} \cos\left(\frac{1}{3}\theta\right)d\theta = 3\sin\left(\frac{1}{3}\theta\right)\Big|_0^{\pi/2} = \frac{3}{2}.$

28. $\int_{\pi/4}^{5\pi/8} \cos 2x\,dx$

SOLUTION $\int_{\pi/4}^{5\pi/8} \cos 2x\,dx = \frac{1}{2}\sin 2x\Big|_{\pi/4}^{5\pi/8} = \frac{1}{2}\sin\frac{5\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2} = -\frac{\sqrt{2}}{4} - \frac{1}{2}.$

29. $\int_{0}^{\pi/6} \sec^2\left(3t - \frac{\pi}{6}\right)dt$

SOLUTION $\int_{0}^{\pi/6} \sec^2\left(3t - \frac{\pi}{6}\right)dt = \frac{1}{3}\tan\left(3t - \frac{\pi}{6}\right)\Big|_0^{\pi/6} = \frac{1}{3}\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right) = \frac{4}{3\sqrt{3}}.$

30. $\int_0^{\pi/6} \sec\theta \tan\theta \, d\theta$

SOLUTION $\int_0^{\pi/6} \sec\theta \tan\theta \, d\theta = \sec\theta \Big|_0^{\pi/6} = \sec\frac{\pi}{6} - \sec 0 = \frac{2\sqrt{3}}{3} - 1.$

31. $\int_{\pi/20}^{\pi/10} \csc 5x \cot 5x \, dx$

SOLUTION $\int_{\pi/20}^{\pi/10} \csc 5x \cot 5x \, dx = -\frac{1}{5}\csc 5x \Big|_{\pi/20}^{\pi/10} = -\frac{1}{5}\left(1 - \sqrt{2}\right) = \frac{1}{5}(\sqrt{2} - 1).$

32. $\int_{\pi/28}^{\pi/14} \csc^2 7y \, dy$

SOLUTION $\int_{\pi/28}^{\pi/14} \csc^2 7y \, dy = -\frac{1}{7}\cot 7y \Big|_{\pi/28}^{\pi/14} = -\frac{1}{7}\cot\frac{\pi}{2} + \frac{1}{7}\cot\frac{\pi}{4} = \frac{1}{7}.$

33. $\int_0^1 e^x \, dx$

SOLUTION $\int_0^1 e^x \, dx = e^x \Big|_0^1 = e - 1.$

34. $\int_3^5 e^{-4x} \, dx$

SOLUTION $\int_3^5 e^{-4x} \, dx = -\frac{1}{4}e^{-4x} \Big|_3^5 = -\frac{1}{4}e^{-20} + \frac{1}{4}e^{-12}.$

35. $\int_0^3 e^{1-6t} \, dt$

SOLUTION $\int_0^3 e^{1-6t} \, dt = -\frac{1}{6}e^{1-6t} \Big|_0^3 = -\frac{1}{6}e^{-17} + \frac{1}{6}e = \frac{1}{6}(e - e^{-17}).$

36. $\int_2^3 e^{4t-3} \, dt$

SOLUTION $\int_2^3 e^{4t-3} \, dt = \frac{1}{4}e^{4t-3} \Big|_2^3 = \frac{1}{4}e^9 - \frac{1}{4}e^5.$

37. $\int_2^{10} \frac{dx}{x}$

SOLUTION $\int_2^{10} \frac{dx}{x} = \ln|x| \Big|_2^{10} = \ln 10 - \ln 2 = \ln 5.$

38. $\int_{-12}^{-4} \frac{dx}{x}$

SOLUTION $\int_{-12}^{-4} \frac{dx}{x} = \ln|x| \Big|_{-12}^{-4} = \ln|-4| - \ln|-12| = \ln\frac{1}{3} = -\ln 3.$

39. $\int_0^1 \frac{dt}{t+1}$

SOLUTION $\int_0^1 \frac{dt}{t+1} = \ln|t+1| \Big|_0^1 = \ln 2 - \ln 1 = \ln 2.$

40. $\int_1^4 \frac{dt}{5t+4}$

SOLUTION $\int_1^4 \frac{dt}{5t+4} = \frac{1}{5}\ln|5t+4| \Big|_1^4 = \frac{1}{5}\ln 24 - \frac{1}{5}\ln 9 = \frac{1}{5}\ln\frac{24}{9}.$

41. $\int_{-2}^0 (3x - 9e^{3x}) \, dx$

SOLUTION $\int_{-2}^0 (3x - 9e^{3x}) \, dx = \left(\frac{3}{2}x^2 - 3e^{3x}\right)\Big|_{-2}^0 = (0 - 3) - (6 - 3e^{-6}) = 3e^{-6} - 9.$

42. $\int_2^6 \left(x + \frac{1}{x}\right) dx$

SOLUTION $\int_2^6 \left(x + \frac{1}{x}\right) dx = \left(\frac{1}{2}x^2 + \ln|x|\right)\Big|_2^6 = (18 + \ln 6) - (2 + \ln 2) = 16 + \ln 3.$

In Exercises 43–48, write the integral as a sum of integrals without absolute values and evaluate.

43. $\int_{-2}^1 |x|\, dx$

SOLUTION

$$\int_{-2}^1 |x|\, dx = \int_{-2}^0 (-x)\, dx + \int_0^1 x\, dx = -\frac{1}{2}x^2\Big|_{-2}^0 + \frac{1}{2}x^2\Big|_0^1 = 0 - \left(-\frac{1}{2}(4)\right) + \frac{1}{2} = \frac{5}{2}.$$

44. $\int_0^5 |3 - x|\, dx$

SOLUTION

$$\begin{aligned}\int_0^5 |3 - x|\, dx &= \int_0^3 (3 - x)\, dx + \int_3^5 (x - 3)\, dx = \left(3x - \frac{1}{2}x^2\right)\Big|_0^3 + \left(\frac{1}{2}x^2 - 3x\right)\Big|_3^5 \\ &= \left(9 - \frac{9}{2}\right) - 0 + \left(\frac{25}{2} - 15\right) - \left(\frac{9}{2} - 9\right) = \frac{13}{2}.\end{aligned}$$

45. $\int_{-2}^3 |x^3|\, dx$

SOLUTION

$$\begin{aligned}\int_{-2}^3 |x^3|\, dx &= \int_{-2}^0 (-x^3)\, dx + \int_0^3 x^3\, dx = -\frac{1}{4}x^4\Big|_{-2}^0 + \frac{1}{4}x^4\Big|_0^3 \\ &= 0 + \frac{1}{4}(-2)^4 + \frac{1}{4}3^4 - 0 = \frac{97}{4}.\end{aligned}$$

46. $\int_0^3 |x^2 - 1|\, dx$

SOLUTION

$$\begin{aligned}\int_0^3 |x^2 - 1|\, dx &= \int_0^1 (1 - x^2)\, dx + \int_1^3 (x^2 - 1)\, dx = \left(x - \frac{1}{3}x^3\right)\Big|_0^1 + \left(\frac{1}{3}x^3 - x\right)\Big|_1^3 \\ &= \left(1 - \frac{1}{3}\right) - 0 + (9 - 3) - \left(\frac{1}{3} - 1\right) = \frac{22}{3}.\end{aligned}$$

47. $\int_0^\pi |\cos x|\, dx$

SOLUTION

$$\int_0^\pi |\cos x|\, dx = \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^\pi (-\cos x)\, dx = \sin x\Big|_0^{\pi/2} - \sin x\Big|_{\pi/2}^\pi = 1 - 0 - (-1 - 0) = 2.$$

48. $\int_0^5 |x^2 - 4x + 3|\, dx$

SOLUTION

$$\begin{aligned}\int_0^5 |x^2 - 4x + 3|\, dx &= \int_0^5 |(x - 3)(x - 1)|\, dx \\ &= \int_0^1 (x^2 - 4x + 3)\, dx + \int_1^3 -(x^2 - 4x + 3)\, dx + \int_3^5 (x^2 - 4x + 3)\, dx\end{aligned}$$

$$= \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_0^1 - \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_1^3 + \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_3^5$$
$$= \left(\frac{1}{3} - 2 + 3\right) - 0 - (9 - 18 + 9) + \left(\frac{1}{3} - 2 + 3\right) + \left(\frac{125}{3} - 50 + 15\right) - (9 - 18 + 9)$$
$$= \frac{28}{3}.$$

In Exercises 49–54, evaluate the integral in terms of the constants.

49. $\int_1^b x^3\,dx$

SOLUTION $\int_1^b x^3\,dx = \frac{1}{4}x^4\Big|_1^b = \frac{1}{4}b^4 - \frac{1}{4}(1)^4 = \frac{1}{4}\left(b^4 - 1\right)$ for any number b.

50. $\int_b^a x^4\,dx$

SOLUTION $\int_b^a x^4\,dx = \frac{1}{5}x^5\Big|_b^a = \frac{1}{5}a^5 - \frac{1}{5}b^5$ for any numbers a, b.

51. $\int_1^b x^5\,dx$

SOLUTION $\int_1^b x^5\,dx = \frac{1}{6}x^6\Big|_1^b = \frac{1}{6}b^6 - \frac{1}{6}(1)^6 = \frac{1}{6}(b^6 - 1)$ for any number b.

52. $\int_{-x}^x (t^3 + t)\,dt$

SOLUTION

$$\int_{-x}^x (t^3 + t)\,dt = \left(\frac{1}{4}t^4 + \frac{1}{2}t^2\right)\Big|_{-x}^x = \left(\frac{1}{4}x^4 + \frac{1}{2}x^2\right) - \left(\frac{1}{4}x^4 + \frac{1}{2}x^2\right) = 0.$$

53. $\int_a^{5a} \frac{dx}{x}$

SOLUTION $\int_a^{5a} \frac{dx}{x} = \ln|x|\Big|_a^{5a} = \ln|5a| - \ln|a| = \ln 5.$

54. $\int_b^{b^2} \frac{dx}{x}$

SOLUTION $\int_b^{b^2} \frac{dx}{x} = \ln|x|\Big|_b^{b^2} = \ln|b^2| - \ln|b| = \ln|b|.$

55. Calculate $\int_{-2}^3 f(x)\,dx$, where

$$f(x) = \begin{cases} 12 - x^2 & \text{for } x \le 2 \\ x^3 & \text{for } x > 2 \end{cases}$$

SOLUTION

$$\int_{-2}^3 f(x)\,dx = \int_{-2}^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_{-2}^2 (12 - x^2)\,dx + \int_2^3 x^3\,dx$$
$$= \left(12x - \frac{1}{3}x^3\right)\Big|_{-2}^2 + \frac{1}{4}x^4\Big|_2^3$$
$$= \left(12(2) - \frac{1}{3}2^3\right) - \left(12(-2) - \frac{1}{3}(-2)^3\right) + \frac{1}{4}3^4 - \frac{1}{4}2^4$$
$$= \frac{128}{3} + \frac{65}{4} = \frac{707}{12}.$$

56. Calculate $\int_0^{2\pi} f(x)\,dx$, where

$$f(x) = \begin{cases} \cos x & \text{for } x \le \pi \\ \cos x - \sin 2x & \text{for } x > \pi \end{cases}$$

SOLUTION

$$\int_0^{2\pi} f(x)\,dx = \int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx = \int_0^{\pi} \cos x\,dx + \int_{\pi}^{2\pi} (\cos x - \sin 2x)\,dx$$

$$= \sin x\Big|_0^{\pi} + \left(\sin x + \frac{1}{2}\cos 2x\right)\Big|_{\pi}^{2\pi}$$

$$= (0-0) + \left(\left(0+\frac{1}{2}\right) - \left(0+\frac{1}{2}\right)\right) = 0.$$

57. Use FTC I to show that $\int_{-1}^{1} x^n\,dx = 0$ if n is an odd whole number. Explain graphically.

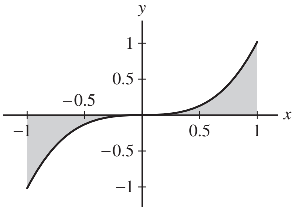
SOLUTION We have

$$\int_{-1}^{1} x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_{-1}^{1} = \frac{(1)^{n+1}}{n+1} - \frac{(-1)^{n+1}}{n+1}.$$

Because n is odd, $n+1$ is even, which means that $(-1)^{n+1} = (1)^{n+1} = 1$. Hence

$$\frac{(1)^{n+1}}{n+1} - \frac{(-1)^{n+1}}{n+1} = \frac{1}{n+1} - \frac{1}{n+1} = 0.$$

Graphically speaking, for an odd function such as x^3 shown here, the positively signed area from $x = 0$ to $x = 1$ cancels the negatively signed area from $x = -1$ to $x = 0$.

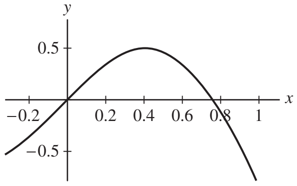


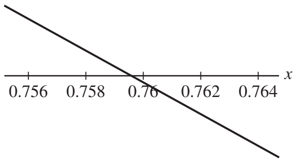
58. CAS Plot the function $f(x) = \sin 3x - x$. Find the positive root of $f(x)$ to three places and use it to find the area under the graph of $f(x)$ in the first quadrant.

SOLUTION The graph of $f(x) = \sin 3x - x$ is shown below at the left. In the figure below at the right, we zoom in on the positive root of $f(x)$ and find that, to three decimal places, this root is approximately $x = 0.760$. The area under the graph of $f(x)$ in the first quadrant is then

$$\int_0^{0.760} (\sin 3x - x)\,dx = \left(-\frac{1}{3}\cos 3x - \frac{1}{2}x^2\right)\Big|_0^{0.760}$$

$$= -\frac{1}{3}\cos(2.28) - \frac{1}{2}(0.760)^2 + \frac{1}{3} \approx 0.262$$





59. Calculate $F(4)$ given that $F(1) = 3$ and $F'(x) = x^2$. *Hint:* Express $F(4) - F(1)$ as a definite integral.

SOLUTION By FTC I,

$$F(4) - F(1) = \int_1^4 x^2\,dx = \frac{4^3 - 1^3}{3} = 21$$

Therefore $F(4) = F(1) + 21 = 3 + 21 = 24$.

60. Calculate $G(16)$, where $dG/dt = t^{-1/2}$ and $G(9) = -5$.

SOLUTION By FTC I,

$$G(16) - G(9) = \int_9^{16} t^{-1/2}\, dt = 2(16^{1/2}) - 2(9^{1/2}) = 2$$

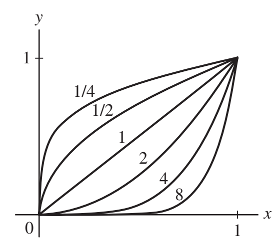
Therefore $G(16) = -5 + 2 = -3$.

61. Does $\int_0^1 x^n\, dx$ get larger or smaller as n increases? Explain graphically.

SOLUTION Let $n \geq 0$ and consider $\int_0^1 x^n\, dx$. (Note: for $n < 0$ the integrand $x^n \to \infty$ as $x \to 0+$, so we exclude this possibility.) Now

$$\int_0^1 x^n\, dx = \left(\frac{1}{n+1}x^{n+1}\right)\Big|_0^1 = \left(\frac{1}{n+1}(1)^{n+1}\right) - \left(\frac{1}{n+1}(0)^{n+1}\right) = \frac{1}{n+1},$$

which decreases as n increases. Recall that $\int_0^1 x^n\, dx$ represents the area between the positive curve $f(x) = x^n$ and the x-axis over the interval [0, 1]. Accordingly, this area gets smaller as n gets larger. This is readily evident in the following graph, which shows curves for several values of n.



62. Show that the area of the shaded parabolic arch in Figure 8 is equal to four-thirds the area of the triangle shown.

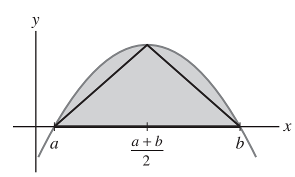


FIGURE 8 Graph of $y = (x - a)(b - x)$.

SOLUTION We first calculate the area of the parabolic arch:

$$\begin{aligned}
\int_a^b (x-a)(b-x)\, dx &= -\int_a^b (x-a)(x-b)\, dx = -\int_a^b (x^2 - ax - bx + ab)\, dx \\
&= -\left(\frac{1}{3}x^3 - \frac{a}{2}x^2 - \frac{b}{2}x^2 + abx\right)\Big|_a^b \\
&= -\frac{1}{6}\left(2x^3 - 3ax^2 - 3bx^2 + 6abx\right)\Big|_a^b \\
&= -\frac{1}{6}\left((2b^3 - 3ab^2 - 3b^3 + 6ab^2) - (2a^3 - 3a^3 - 3ba^2 + 6a^2b)\right) \\
&= -\frac{1}{6}\left((-b^3 + 3ab^2) - (-a^3 + 3a^2b)\right) \\
&= -\frac{1}{6}\left(a^3 + 3ab^2 - 3a^2b - b^3\right) = \frac{1}{6}(b-a)^3.
\end{aligned}$$

The indicated triangle has a base of length $b - a$ and a height of

$$\left(\frac{a+b}{2} - a\right)\left(b - \frac{a+b}{2}\right) = \left(\frac{b-a}{2}\right)^2.$$

Thus, the area of the triangle is

$$\frac{1}{2}(b-a)\left(\frac{b-a}{2}\right)^2 = \frac{1}{8}(b-a)^3.$$

Finally, we note that

$$\frac{1}{6}(b-a)^3 = \frac{4}{3}\cdot\frac{1}{8}(b-a)^3,$$

as required.

Further Insights and Challenges

63. Prove a famous result of Archimedes (generalizing Exercise 62): For $r < s$, the area of the shaded region in Figure 9 is equal to four-thirds the area of triangle $\triangle ACE$, where C is the point on the parabola at which the tangent line is parallel to secant line $\overline{AE}$.

(a) Show that C has x-coordinate $(r+s)/2$.

(b) Show that $ABDE$ has area $(s-r)^3/4$ by viewing it as a parallelogram of height $s-r$ and base of length $\overline{CF}$.

(c) Show that $\triangle ACE$ has area $(s-r)^3/8$ by observing that it has the same base and height as the parallelogram.

(d) Compute the shaded area as the area under the graph minus the area of a trapezoid, and prove Archimedes' result.

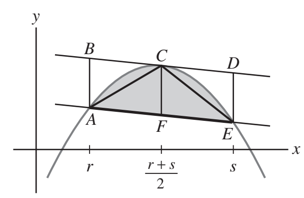


FIGURE 9 Graph of $f(x) = (x-a)(b-x)$.

SOLUTION

(a) The slope of the secant line $\overline{AE}$ is

$$\frac{f(s)-f(r)}{s-r} = \frac{(s-a)(b-s)-(r-a)(b-r)}{s-r} = a+b-(r+s)$$

and the slope of the tangent line along the parabola is

$$f'(x) = a+b-2x.$$

If C is the point on the parabola at which the tangent line is parallel to the secant line $\overline{AE}$, then its x-coordinate must satisfy

$$a+b-2x = a+b-(r+s) \qquad \text{or} \qquad x = \frac{r+s}{2}.$$

(b) Parallelogram $ABDE$ has height $s-r$ and base of length $\overline{CF}$. Since the equation of the secant line $\overline{AE}$ is

$$y = [a+b-(r+s)]\,(x-r) + (r-a)(b-r),$$

the length of the segment $\overline{CF}$ is

$$\left(\frac{r+s}{2}-a\right)\left(b-\frac{r+s}{2}\right) - [a+b-(r+s)]\left(\frac{r+s}{2}-r\right) - (r-a)(b-r) = \frac{(s-r)^2}{4}.$$

Thus, the area of $ABDE$ is $\frac{(s-r)^3}{4}$.

(c) Triangle ACE is comprised of ΔACF and ΔCEF. Each of these smaller triangles has height $\frac{s-r}{2}$ and base of length $\frac{(s-r)^2}{4}$. Thus, the area of ΔACE is

$$\frac{1}{2}\frac{s-r}{2}\cdot\frac{(s-r)^2}{4} + \frac{1}{2}\frac{s-r}{2}\cdot\frac{(s-r)^2}{4} = \frac{(s-r)^3}{8}.$$

(d) The area under the graph of the parabola between $x=r$ and $x=s$ is

$$\begin{aligned}\int_r^s (x-a)(b-x)\,dx &= \left(-abx + \frac{1}{2}(a+b)x^2 - \frac{1}{3}x^3\right)\Big|_r^s \\ &= -abs + \frac{1}{2}(a+b)s^2 - \frac{1}{3}s^3 + abr - \frac{1}{2}(a+b)r^2 + \frac{1}{3}r^3 \\ &= ab(r-s) + \frac{1}{2}(a+b)(s-r)(s+r) + \frac{1}{3}(r-s)(r^2+rs+s^2),\end{aligned}$$

while the area of the trapezoid under the shaded region is

$$\begin{aligned}&\frac{1}{2}(s-r)\,[(s-a)(b-s)+(r-a)(b-r)]\\&\quad=\frac{1}{2}(s-r)\left[-2ab+(a+b)(r+s)-r^2-s^2\right]\\&\quad=ab(r-s)+\frac{1}{2}(a+b)(s-r)(r+s)+\frac{1}{2}(r-s)(r^2+s^2).\end{aligned}$$

Thus, the area of the shaded region is

$$(r-s)\left(\frac{1}{3}r^2+\frac{1}{3}rs+\frac{1}{3}s^2-\frac{1}{2}r^2-\frac{1}{2}s^2\right)=(s-r)\left(\frac{1}{6}r^2-\frac{1}{3}rs+\frac{1}{6}s^2\right)=\frac{1}{6}(s-r)^3,$$

which is four-thirds the area of the triangle ACE.

64. **(a)** Apply the Comparison Theorem (Theorem 5 in Section 5.2) to the inequality $\sin x \le x$ (valid for $x \ge 0$) to prove that

$$1-\frac{x^2}{2}\le \cos x\le 1$$

(b) Apply it again to prove that

$$x-\frac{x^3}{6}\le \sin x\le x \quad (\text{for } x\ge 0)$$

(c) Verify these inequalities for $x = 0.3$.

SOLUTION

(a) We have $\int_0^x \sin t\,dt = -\cos t\Big|_0^x = -\cos x + 1$ and $\int_0^x t\,dt = \frac{1}{2}t^2\Big|_0^x = \frac{1}{2}x^2$. Hence

$$-\cos x+1\le\frac{x^2}{2}.$$

Solving, this gives $\cos x \ge 1 - \frac{x^2}{2}$. $\cos x \le 1$ follows automatically.

(b) The previous part gives us $1 - \frac{t^2}{2} \le \cos t \le 1$, for $t > 0$. Theorem 5 gives us, after integrating over the interval $[0, x]$,

$$x-\frac{x^3}{6}\le\sin x\le x.$$

(c) Substituting $x = 0.3$ into the inequalities obtained in (a) and (b) yields

$$0.955\le 0.955336489\le 1 \qquad\text{and}\qquad 0.2955\le 0.2955202069\le 0.3,$$

respectively.

65. Use the method of Exercise 64 to prove that

$$\begin{aligned}1-\frac{x^2}{2}\le\cos x\le 1-\frac{x^2}{2}+\frac{x^4}{24}\\x-\frac{x^3}{6}\le\sin x\le x-\frac{x^3}{6}+\frac{x^5}{120}\quad(\text{for } x\ge 0)\end{aligned}$$

Verify these inequalities for $x = 0.1$. Why have we specified $x \ge 0$ for $\sin x$ but not for $\cos x$?

SOLUTION By Exercise 64, $t - \frac{1}{6}t^3 \le \sin t \le t$ for $t > 0$. Integrating this inequality over the interval $[0, x]$, and then solving for $\cos x$, yields:

$$\begin{aligned}\frac{1}{2}x^2-\frac{1}{24}x^4\le 1-\cos x\le\frac{1}{2}x^2\\1-\frac{1}{2}x^2\le\cos x\le 1-\frac{1}{2}x^2+\frac{1}{24}x^4.\end{aligned}$$

These inequalities apply for $x \ge 0$. Since $\cos x$, $1 - \frac{x^2}{2}$, and $1 - \frac{x^2}{2} + \frac{x^4}{24}$ are all even functions, they also apply for $x \le 0$.

Having established that

$$1 - \frac{t^2}{2} \le \cos t \le 1 - \frac{t^2}{2} + \frac{t^4}{24},$$

for all $t \ge 0$, we integrate over the interval $[0, x]$, to obtain:

$$x - \frac{x^3}{6} \le \sin x \le x - \frac{x^3}{6} + \frac{x^5}{120}.$$

The functions $\sin x$, $x - \frac{1}{6}x^3$ and $x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ are all odd functions, so the inequalities are reversed for $x < 0$.

Evaluating these inequalities at $x = 0.1$ yields

$$0.995000000 \le 0.995004165 \le 0.995004167$$

$$0.0998333333 \le 0.0998334166 \le 0.0998334167,$$

both of which are true.

66. Calculate the next pair of inequalities for $\sin x$ and $\cos x$ by integrating the results of Exercise 65. Can you guess the general pattern?

SOLUTION Integrating

$$t - \frac{t^3}{6} \le \sin t \le t - \frac{t^3}{6} + \frac{t^5}{120} \qquad (\text{for } t \ge 0)$$

over the interval $[0, x]$ yields

$$\frac{x^2}{2} - \frac{x^4}{24} \le 1 - \cos x \le \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720}.$$

Solving for $\cos x$ and yields

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \le \cos x \le 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Replacing each x by t and integrating over the interval $[0, x]$ produces

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \le \sin x \le x - \frac{x^3}{6} + \frac{x^5}{120}.$$

To see the pattern, it is best to compare consecutive inequalities for $\sin x$ and those for $\cos x$:

$$0 \le \sin x \le x$$

$$x - \frac{x^3}{6} \le \sin x \le x$$

$$x - \frac{x^3}{6} \le \sin x \le x - \frac{x^3}{6} + \frac{x^5}{120}.$$

Each iteration adds an additional term. Looking at the highest order terms, we get the following pattern:

$$0$$

$$x$$

$$-\frac{x^3}{6} = -\frac{x^3}{3!}$$

$$\frac{x^5}{5!}$$

We guess that the leading term of the polynomials are of the form

$$(-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Similarly, for $\cos x$, the leading terms of the polynomials in the inequality are of the form

$$(-1)^n \frac{x^{2n}}{(2n)!}.$$

67. Use FTC I to prove that if $|f'(x)| \le K$ for $x \in [a, b]$, then $|f(x) - f(a)| \le K|x - a|$ for $x \in [a, b]$.

SOLUTION Let $a > b$ be real numbers, and let $f(x)$ be such that $|f'(x)| \le K$ for $x \in [a, b]$. By FTC,

$$\int_a^x f'(t)\,dt = f(x) - f(a).$$

Since $f'(x) \ge -K$ for all $x \in [a, b]$, we get:

$$f(x) - f(a) = \int_a^x f'(t)\, dt \ge -K(x - a).$$

Since $f'(x) \le K$ for all $x \in [a, b]$, we get:

$$f(x) - f(a) = \int_a^x f'(t)\, dt \le K(x - a).$$

Combining these two inequalities yields

$$-K(x - a) \le f(x) - f(a) \le K(x - a),$$

so that, by definition,

$$|f(x) - f(a)| \le K|x - a|.$$

68. **(a)** Use Exercise 67 to prove that $|\sin a - \sin b| \le |a - b|$ for all a, b.
(b) Let $f(x) = \sin(x + a) - \sin x$. Use part (a) to show that the graph of $f(x)$ lies between the horizontal lines $y = \pm a$.
(c) GU Plot $f(x)$ and the lines $y = \pm a$ to verify (b) for $a = 0.5$ and $a = 0.2$.

SOLUTION
(a) Let $f(x) = \sin x$, so that $f'(x) = \cos x$, and

$$|f'(x)| \le 1$$

for all x. From Exercise 67, we get:

$$|\sin a - \sin b| \le |a - b|.$$

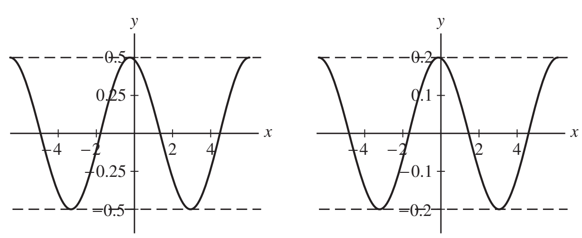
(b) Let $f(x) = \sin(x + a) - \sin(x)$. Applying (a), we get the inequality:

$$|f(x)| = |\sin(x + a) - \sin(x)| \le |(x + a - x)| = |a|.$$

This is equivalent, by definition, to the two inequalities:

$$-a \le \sin(x + a) - \sin(x) \le a.$$

(c) The plots of $y = \sin(x + 0.5) - \sin(x)$ and of $y = \sin(x + 0.2) - \sin(x)$ are shown below. The inequality is satisfied in both plots.



5.4 The Fundamental Theorem of Calculus, Part II

Preliminary Questions

1. Let $G(x) = \int_4^x \sqrt{t^3 + 1}\, dt$.
(a) Is the FTC needed to calculate $G(4)$?
(b) Is the FTC needed to calculate $G'(4)$?

SOLUTION
(a) No. $G(4) = \int_4^4 \sqrt{t^3 + 1}\, dt = 0$.
(b) Yes. By the FTC II, $G'(x) = \sqrt{x^3 + 1}$, so $G'(4) = \sqrt{65}$.

2. Which of the following is an antiderivative $F(x)$ of $f(x) = x^2$ satisfying $F(2) = 0$?

(a) $\int_2^x 2t\, dt$ **(b)** $\int_0^2 t^2\, dt$ **(c)** $\int_2^x t^2\, dt$

SOLUTION The correct answer is **(c)**: $\int_2^x t^2\, dt$.

3. Does every continuous function have an antiderivative? Explain.

SOLUTION Yes. All continuous functions have an antiderivative, namely $\int_a^x f(t)\,dt$.

4. Let $G(x) = \int_4^{x^3} \sin t\,dt$. Which of the following statements are correct?

(a) $G(x)$ is the composite function $\sin(x^3)$.

(b) $G(x)$ is the composite function $A(x^3)$, where

$$A(x) = \int_4^x \sin(t)\,dt$$

(c) $G(x)$ is too complicated to differentiate.

(d) The Product Rule is used to differentiate $G(x)$.

(e) The Chain Rule is used to differentiate $G(x)$.

(f) $G'(x) = 3x^2 \sin(x^3)$.

SOLUTION Statements **(b)**, **(e)**, and **(f)** are correct.

Exercises

1. Write the area function of $f(x) = 2x + 4$ with lower limit $a = -2$ as an integral and find a formula for it.

SOLUTION Let $f(x) = 2x + 4$. The area function with lower limit $a = -2$ is

$$A(x) = \int_a^x f(t)\,dt = \int_{-2}^x (2t + 4)\,dt.$$

Carrying out the integration, we find

$$\int_{-2}^x (2t + 4)\,dt = (t^2 + 4t)\Big|_{-2}^x = (x^2 + 4x) - ((-2)^2 + 4(-2)) = x^2 + 4x + 4$$

or $(x + 2)^2$. Therefore, $A(x) = (x + 2)^2$.

2. Find a formula for the area function of $f(x) = 2x + 4$ with lower limit $a = 0$.

SOLUTION The area function for $f(x) = 2x + 4$ with lower limit $a = 0$ is given by

$$A(x) = \int_0^x (2t + 4)\,dt = (t^2 + 4t)\Big|_0^x = x^2 + 4x.$$

3. Let $G(x) = \int_1^x (t^2 - 2)\,dt$. Calculate $G(1)$, $G'(1)$ and $G'(2)$. Then find a formula for $G(x)$.

SOLUTION Let $G(x) = \int_1^x (t^2 - 2)\,dt$. Then $G(1) = \int_1^1 (t^2 - 2)\,dt = 0$. Moreover, $G'(x) = x^2 - 2$, so that $G'(1) = -1$ and $G'(2) = 2$. Finally,

$$G(x) = \int_1^x (t^2 - 2)\,dt = \left(\frac{1}{3}t^3 - 2t\right)\Big|_1^x = \left(\frac{1}{3}x^3 - 2x\right) - \left(\frac{1}{3}(1)^3 - 2(1)\right) = \frac{1}{3}x^3 - 2x + \frac{5}{3}.$$

4. Find $F(0)$, $F'(0)$, and $F'(3)$, where $F(x) = \int_0^x \sqrt{t^2 + t}\,dt$.

SOLUTION By definition, $F(0) = \int_0^0 \sqrt{t^2 + t}\,dt = 0$. By FTC, $F'(x) = \sqrt{x^2 + x}$, so that $F'(0) = \sqrt{0^2 + 0} = 0$ and $F'(3) = \sqrt{3^2 + 3} = \sqrt{12} = 2\sqrt{3}$.

5. Find $G(1)$, $G'(0)$, and $G'(\pi/4)$, where $G(x) = \int_1^x \tan t\,dt$.

SOLUTION By definition, $G(1) = \int_1^1 \tan t\,dt = 0$. By FTC, $G'(x) = \tan x$, so that $G'(0) = \tan 0 = 0$ and $G'(\frac{\pi}{4}) = \tan \frac{\pi}{4} = 1$.

6. Find $H(-2)$ and $H'(-2)$, where $H(x) = \int_{-2}^x \frac{du}{u^2 + 1}$.

SOLUTION By definition, $H(-2) = \int_{-2}^{-2} \frac{du}{u^2 + 1} = 0$. By FTC, $H'(x) = \frac{1}{x^2 + 1}$, so $H'(-2) = \frac{1}{5}$.

In Exercises 7–16, find formulas for the functions represented by the integrals.

7. $\int_2^x u^4\,du$

SOLUTION $F(x) = \int_2^x u^4\,du = \frac{1}{5}u^5\Big|_2^x = \frac{1}{5}x^5 - \frac{32}{5}.$

8. $\int_2^x (12t^2 - 8t)\,dt$

SOLUTION $F(x) = \int_2^x (12t^2 - 8t)\,dt = (4t^3 - 4t^2)\Big|_2^x = 4x^3 - 4x^2 - 16.$

9. $\int_0^x \sin u\,du$

SOLUTION $F(x) = \int_0^x \sin u\,du = (-\cos u)\Big|_0^x = 1 - \cos x.$

10. $\int_{-\pi/4}^x \sec^2\theta\,d\theta$

SOLUTION $F(x) = \int_{-\pi/4}^x \sec^2\theta\,d\theta = \tan\theta\Big|_{-\pi/4}^x = \tan x - \tan(-\pi/4) = \tan x + 1.$

11. $\int_4^x e^{3u}\,du$

SOLUTION $F(x) = \int_4^x e^{3u}\,du = \frac{1}{3}e^{3u}\Big|_4^x = \frac{1}{3}e^{3x} - \frac{1}{3}e^{12}.$

12. $\int_x^0 e^{-t}\,dt$

SOLUTION $F(x) = \int_x^0 e^{-t}\,dt = -e^{-t}\Big|_x^0 = -1 + e^{-x}.$

13. $\int_1^{x^2} t\,dt$

SOLUTION $F(x) = \int_1^{x^2} t\,dt = \frac{1}{2}t^2\Big|_1^{x^2} = \frac{1}{2}x^4 - \frac{1}{2}.$

14. $\int_{x/2}^{x/4} \sec^2 u\,du$

SOLUTION $F(x) = \int_{x/2}^{x/4} \sec^2 u\,du = \tan u\Big|_{x/2}^{x/4} = \tan\frac{x}{4} - \tan\frac{x}{2}.$

15. $\int_{3x}^{9x+2} e^{-u}\,du$

SOLUTION $F(x) = \int_{3x}^{9x+2} e^{-u}\,du = -e^{-u}\Big|_{3x}^{9x+2} = -e^{-9x-2} + e^{-3x}.$

16. $\int_2^{\sqrt{x}} \frac{dt}{t}$

SOLUTION $\int_2^{\sqrt{x}} \frac{dt}{t} = \ln|t|\Big|_2^{\sqrt{x}} = \ln\sqrt{x} - \ln 2 = \frac{1}{2}\ln x - \ln 2.$

In Exercises 17–20, express the antiderivative $F(x)$ of $f(x)$ satisfying the given initial condition as an integral.

17. $f(x) = \sqrt{x^3 + 1}, \quad F(5) = 0$

SOLUTION The antiderivative $F(x)$ of $\sqrt{x^3+1}$ satisfying $F(5) = 0$ is $F(x) = \int_5^x \sqrt{t^3+1}\,dt.$

18. $f(x) = \frac{x+1}{x^2+9}, \quad F(7) = 0$

SOLUTION The antiderivative $F(x)$ of $f(x) = \frac{x+1}{x^2+9}$ satisfying $F(7) = 0$ is $F(x) = \int_7^x \frac{t+1}{t^2+9}\,dt.$

19. $f(x) = \sec x, \quad F(0) = 0$

SOLUTION The antiderivative $F(x)$ of $f(x) = \sec x$ satisfying $F(0) = 0$ is $F(x) = \int_0^x \sec t\,dt.$

20. $f(x) = e^{-x^2}, \quad F(-4) = 0$

SOLUTION The antiderivative $F(x)$ of $f(x) = e^{-x^2}$ satisfying $F(-4) = 0$ is

$$F(x) = \int_{-4}^x e^{-t^2}\,dt.$$

In Exercises 21–24, calculate the derivative.

21. $\frac{d}{dx}\int_0^x (t^5 - 9t^3)\,dt$

SOLUTION By FTC II, $\frac{d}{dx}\int_0^x (t^5 - 9t^3)\,dt = x^5 - 9x^3$.

22. $\frac{d}{d\theta}\int_1^\theta \cot u\,du$

SOLUTION By FTC II, $\frac{d}{d\theta}\int_1^\theta \cot u\,du = \cot\theta$.

23. $\frac{d}{dt}\int_{100}^t \sec(5x - 9)\,dx$

SOLUTION By FTC II, $\frac{d}{dt}\int_{100}^t \sec(5x - 9)\,dx = \sec(5t - 9)$.

24. $\frac{d}{ds}\int_{-2}^s \tan\left(\frac{1}{1+u^2}\right)du$

SOLUTION By FTC II, $\frac{d}{ds}\int_{-2}^s \tan\left(\frac{1}{1+u^2}\right)du = \tan\left(\frac{1}{1+s^2}\right)$.

25. Let $A(x) = \int_0^x f(t)\,dt$ for $f(x)$ in Figure 8.

(a) Calculate $A(2)$, $A(3)$, $A'(2)$, and $A'(3)$.

(b) Find formulas for $A(x)$ on $[0, 2]$ and $[2, 4]$ and sketch the graph of $A(x)$.

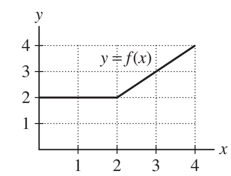


FIGURE 8

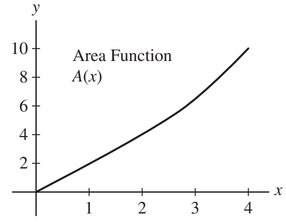
SOLUTION

(a) $A(2) = 2 \cdot 2 = 4$, the area under $f(x)$ from $x = 0$ to $x = 2$, while $A(3) = 2 \cdot 3 + \frac{1}{2} = 6.5$, the area under $f(x)$ from $x = 0$ to $x = 3$. By the FTC, $A'(x) = f(x)$ so $A'(2) = f(2) = 2$ and $A'(3) = f(3) = 3$.

(b) For each $x \in [0, 2]$, the region under the graph of $y = f(x)$ is a rectangle of length x and height 2; for each $x \in [2, 4]$, the region is comprised of a square of side length 2 and a trapezoid of height $x - 2$ and bases 2 and x. Hence,

$$A(x) = \begin{cases} 2x, & 0 \le x < 2 \\ \frac{1}{2}x^2 + 2, & 2 \le x \le 4 \end{cases}$$

A graph of the area function $A(x)$ is shown below.



26. Make a rough sketch of the graph of $A(x) = \int_0^x g(t)\,dt$ for $g(x)$ in Figure 9.

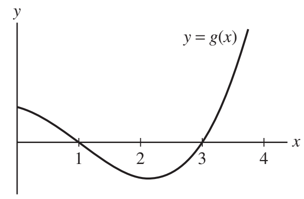
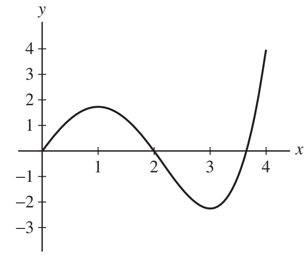


FIGURE 9

SOLUTION The graph of $y = g(x)$ lies above the x-axis over the interval [0, 1], below the x-axis over [1, 3], and above the x-axis over [3, 4]. The corresponding area function should therefore be increasing on (0, 1), decreasing on (1, 3) and increasing on (3, 4). Further, it appears from Figure 9 that the local minimum of the area function at $x = 3$ should be negative. One possible graph of the area function is the following.



27. Verify: $\int_0^x |t|\,dt = \frac{1}{2}x|x|$. *Hint:* Consider $x \geq 0$ and $x \leq 0$ separately.

SOLUTION Let $f(t) = |t| = \begin{cases} t & \text{for } t \geq 0 \\ -t & \text{for } t < 0 \end{cases}$. Then

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} \int_0^x t\,dt & \text{for } x \geq 0 \\ \int_0^x -t\,dt & \text{for } x < 0 \end{cases} = \begin{cases} \frac{1}{2}t^2\Big|_0^x = \frac{1}{2}x^2 & \text{for } x \geq 0 \\ \left(-\frac{1}{2}t^2\right)\Big|_0^x = -\frac{1}{2}x^2 & \text{for } x < 0 \end{cases}$$

For $x \geq 0$, we have $F(x) = \frac{1}{2}x^2 = \frac{1}{2}x\,|x|$ since $|x| = x$, while for $x < 0$, we have $F(x) = -\frac{1}{2}x^2 = \frac{1}{2}x\,|x|$ since $|x| = -x$. Therefore, for all real x we have $F(x) = \frac{1}{2}x\,|x|$.

28. Find $G'(1)$, where $G(x) = \int_0^{x^2} \sqrt{t^3 + 3}\,dt$.

SOLUTION By combining the Chain Rule and FTC, $G'(x) = \sqrt{x^6 + 3} \cdot 2x$, so $G'(1) = \sqrt{1 + 3} \cdot 2 = 4$.

In Exercises 29–34, calculate the derivative.

29. $\frac{d}{dx}\int_0^{x^2} \frac{t\,dt}{t+1}$

SOLUTION By the Chain Rule and the FTC, $\frac{d}{dx}\int_0^{x^2} \frac{t\,dt}{t+1} = \frac{x^2}{x^2+1} \cdot 2x = \frac{2x^3}{x^2+1}$.

30. $\frac{d}{dx}\int_1^{1/x} \cos^3 t\,dt$

SOLUTION By the Chain Rule and the FTC, $\frac{d}{dx}\int_1^{1/x} \cos^3 t\,dt = \cos^3\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}\cos^3\left(\frac{1}{x}\right)$.

31. $\frac{d}{ds}\int_{-6}^{\cos s} u^4\,du$

SOLUTION By the Chain Rule and the FTC, $\frac{d}{ds}\int_{-6}^{\cos s} u^4\,du = \cos^4 s(-\sin s) = -\cos^4 s \sin s$.

32. $\frac{d}{dx}\int_{x^2}^{x^4} \sqrt{t}\,dt$

Hint for Exercise 32: $F(x) = A(x^4) - A(x^2)$.

SOLUTION Let

$$F(x) = \int_{x^2}^{x^4} \sqrt{t}\,dt = \int_0^{x^4} \sqrt{t}\,dt - \int_0^{x^2} \sqrt{t}\,dt.$$

Applying the Chain Rule combined with FTC, we have

$$F'(x) = \sqrt{x^4} \cdot 4x^3 - \sqrt{x^2} \cdot 2x = 4x^5 - 2x\,|x|.$$

33. $\dfrac{d}{dx}\displaystyle\int_{\sqrt{x}}^{x^2} \tan t\,dt$

SOLUTION Let

$$G(x) = \int_{\sqrt{x}}^{x^2} \tan t\,dt = \int_0^{x^2} \tan t\,dt - \int_0^{\sqrt{x}} \tan t\,dt.$$

Applying the Chain Rule combined with FTC twice, we have

$$G'(x) = \tan(x^2)\cdot 2x - \tan(\sqrt{x})\cdot \frac{1}{2}x^{-1/2} = 2x\tan(x^2) - \frac{\tan(\sqrt{x})}{2\sqrt{x}}.$$

34. $\dfrac{d}{du}\displaystyle\int_{-u}^{3u} \sqrt{x^2+1}\,dx$

SOLUTION Let

$$G(x) = \int_{-u}^{3u} \sqrt{x^2+1}\,dx = \int_0^{3u} \sqrt{x^2+1}\,dx - \int_0^{-u} \sqrt{x^2+1}\,dx.$$

Applying the Chain Rule combined with FTC twice, we have

$$G'(x) = 3\sqrt{9u^2+1} + \sqrt{u^2+1}.$$

In Exercises 35–38, with $f(x)$ as in Figure 10 let

$$A(x) = \int_0^x f(t)\,dt \qquad \textit{and} \qquad B(x) = \int_2^x f(t)\,dt.$$

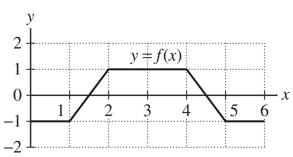


FIGURE 10

35. Find the min and max of $A(x)$ on $[0, 6]$.

SOLUTION The minimum values of $A(x)$ on $[0, 6]$ occur where $A'(x) = f(x)$ goes from negative to positive. This occurs at one place, where $x = 1.5$. The minimum value of $A(x)$ is therefore $A(1.5) = -1.25$. The maximum values of $A(x)$ on $[0, 6]$ occur where $A'(x) = f(x)$ goes from positive to negative. This occurs at one place, where $x = 4.5$. The maximum value of $A(x)$ is therefore $A(4.5) = 1.25$.

36. Find the min and max of $B(x)$ on $[0, 6]$.

SOLUTION The minimum values of $B(x)$ on $[0, 6]$ occur where $B'(x) = f(x)$ goes from negative to positive. This occurs at one place, where $x = 1.5$. The minimum value of $A(x)$ is therefore $B(1.5) = -0.25$. The maximum values of $B(x)$ on $[0, 6]$ occur where $B'(x) = f(x)$ goes from positive to negative. This occurs at one place, where $x = 4.5$. The maximum value of $B(x)$ is therefore $B(4.5) = 2.25$.

37. Find formulas for $A(x)$ and $B(x)$ valid on $[2, 4]$.

SOLUTION On the interval $[2, 4]$, $A'(x) = B'(x) = f(x) = 1$. $A(2) = \int_0^2 f(t)\,dt = -1$ and $B(2) = \int_2^2 f(t)\,dt = 0$. Hence $A(x) = (x-2) - 1$ and $B(x) = (x-2)$.

38. Find formulas for $A(x)$ and $B(x)$ valid on $[4, 5]$.

SOLUTION On the interval $[4, 5]$, $A'(x) = B'(x) = f(x) = -2(x-4.5) = 9 - 2x$. $A(4) = \int_0^4 f(t)\,dt = 1$ and $B(4) = \int_2^4 f(t)\,dt = 2$. Hence $A(x) = 9x - x^2 - 19$ and $B(x) = 9x - x^2 - 18$.

39. Let $A(x) = \int_0^x f(t)\,dt$, with $f(x)$ as in Figure 11.

(a) Does $A(x)$ have a local maximum at P?

(b) Where does $A(x)$ have a local minimum?

(c) Where does $A(x)$ have a local maximum?
(d) True or false? $A(x) < 0$ for all x in the interval shown.

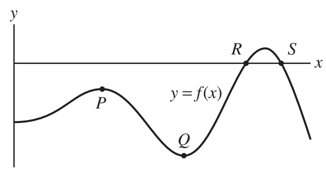


FIGURE 11 Graph of $f(x)$.

SOLUTION

(a) In order for $A(x)$ to have a local maximum, $A'(x) = f(x)$ must transition from positive to negative. As this does not happen at P, $A(x)$ does not have a local maximum at P.
(b) $A(x)$ will have a local minimum when $A'(x) = f(x)$ transitions from negative to positive. This happens at R, so $A(x)$ has a local minimum at R.
(c) $A(x)$ will have a local maximum when $A'(x) = f(x)$ transitions from positive to negative. This happens at S, so $A(x)$ has a local maximum at S.
(d) It is true that $A(x) < 0$ on I since the signed area from 0 to x is clearly always negative from the figure.

40. Determine $f(x)$, assuming that $\int_0^x f(t)\,dt = x^2 + x$.

SOLUTION Let $F(x) = \int_0^x f(t)\,dt = x^2 + x$. Then $F'(x) = f(x) = 2x + 1$.

41. Determine the function $g(x)$ and all values of c such that

$$\int_c^x g(t)\,dt = x^2 + x - 6$$

SOLUTION By the FTC II we have

$$g(x) = \frac{d}{dx}(x^2 + x - 6) = 2x + 1$$

and therefore,

$$\int_c^x g(t)\,dt = x^2 + x - (c^2 + c)$$

We must choose c so that $c^2 + c = 6$. We can take $c = 2$ or $c = -3$.

42. Find $a \le b$ such that $\int_a^b (x^2 - 9)\,dx$ has minimal value.

SOLUTION Let a be given, and let $F_a(x) = \int_a^x (t^2 - 9)\,dt$. Then $F_a'(x) = x^2 - 9$, and the critical points are $x = \pm 3$. Because $F_a''(-3) = -6$ and $F_a''(3) = 6$, we see that $F_a(x)$ has a minimum at $x = 3$. Now, we find a minimizing $\int_a^3 (x^2 - 9)\,dx$. Let $G(x) = \int_x^3 (x^2 - 9)\,dx$. Then $G'(x) = -(x^2 - 9)$, yielding critical points $x = 3$ or $x = -3$. With $x = -3$,

$$G(-3) = \int_{-3}^3 (x^2 - 9)\,dx = \left(\frac{1}{3}x^3 - 9x\right)\Big|_{-3}^3 = -36.$$

With $x = 3$,

$$G(3) = \int_3^3 (x^2 - 9)\,dx = 0.$$

Hence $a = -3$ and $b = 3$ are the values minimizing $\int_a^b (x^2 - 9)\,dx$.

In Exercises 43 and 44, let $A(x) = \int_a^x f(t)\,dt$.

43. **Area Functions and Concavity** Explain why the following statements are true. Assume $f(x)$ is differentiable.
(a) If c is an inflection point of $A(x)$, then $f'(c) = 0$.
(b) $A(x)$ is concave up if $f(x)$ is increasing.
(c) $A(x)$ is concave down if $f(x)$ is decreasing.

SOLUTION

(a) If $x = c$ is an inflection point of $A(x)$, then $A''(c) = f'(c) = 0$.

(b) If $A(x)$ is concave up, then $A''(x) > 0$. Since $A(x)$ is the area function associated with $f(x)$, $A'(x) = f(x)$ by FTC II, so $A''(x) = f'(x)$. Therefore $f'(x) > 0$, so $f(x)$ is increasing.

(c) If $A(x)$ is concave down, then $A''(x) < 0$. Since $A(x)$ is the area function associated with $f(x)$, $A'(x) = f(x)$ by FTC II, so $A''(x) = f'(x)$. Therefore, $f'(x) < 0$ and so $f(x)$ is decreasing.

44. Match the property of $A(x)$ with the corresponding property of the graph of $f(x)$. Assume $f(x)$ is differentiable.

Area function $A(x)$

(a) $A(x)$ is decreasing.

(b) $A(x)$ has a local maximum.

(c) $A(x)$ is concave up.

(d) $A(x)$ goes from concave up to concave down.

Graph of $f(x)$

(i) Lies below the x-axis.

(ii) Crosses the x-axis from positive to negative.

(iii) Has a local maximum.

(iv) $f(x)$ is increasing.

SOLUTION Let $A(x) = \int_a^x f(t)\,dt$ be an area function of $f(x)$. Then $A'(x) = f(x)$ and $A''(x) = f'(x)$.

(a) $A(x)$ is decreasing when $A'(x) = f(x) < 0$, i.e., when $f(x)$ lies below the x-axis. This is choice **(i)**.

(b) $A(x)$ has a local maximum (at x_0) when $A'(x) = f(x)$ changes sign from + to 0 to − as x increases through x_0, i.e., when $f(x)$ crosses the x-axis from positive to negative. This is choice **(ii)**.

(c) $A(x)$ is concave up when $A''(x) = f'(x) > 0$, i.e., when $f(x)$ is increasing. This corresponds to choice **(iv)**.

(d) $A(x)$ goes from concave up to concave down (at x_0) when $A''(x) = f'(x)$ changes sign from + to 0 to − as x increases through x_0, i.e., when $f(x)$ has a local maximum at x_0. This is choice **(iii)**.

45. Let $A(x) = \displaystyle\int_0^x f(t)\,dt$, with $f(x)$ as in Figure 12. Determine:

(a) The intervals on which $A(x)$ is increasing and decreasing

(b) The values x where $A(x)$ has a local min or max

(c) The inflection points of $A(x)$

(d) The intervals where $A(x)$ is concave up or concave down

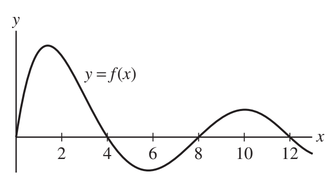


FIGURE 12

SOLUTION

(a) $A(x)$ is increasing when $A'(x) = f(x) > 0$, which corresponds to the intervals $(0, 4)$ and $(8, 12)$. $A(x)$ is decreasing when $A'(x) = f(x) < 0$, which corresponds to the intervals $(4, 8)$ and $(12, \infty)$.

(b) $A(x)$ has a local minimum when $A'(x) = f(x)$ changes from − to +, corresponding to $x = 8$. $A(x)$ has a local maximum when $A'(x) = f(x)$ changes from + to −, corresponding to $x = 4$ and $x = 12$.

(c) Inflection points of $A(x)$ occur where $A''(x) = f'(x)$ changes sign, or where f changes from increasing to decreasing or vice versa. Consequently, $A(x)$ has inflection points at $x = 2$, $x = 6$, and $x = 10$.

(d) $A(x)$ is concave up when $A''(x) = f'(x)$ is positive or $f(x)$ is increasing, which corresponds to the intervals $(0, 2)$ and $(6, 10)$. Similarly, $A(x)$ is concave down when $f(x)$ is decreasing, which corresponds to the intervals $(2, 6)$ and $(10, \infty)$.

46. Let $f(x) = x^2 - 5x - 6$ and $F(x) = \displaystyle\int_0^x f(t)\,dt$.

(a) Find the critical points of $F(x)$ and determine whether they are local minima or local maxima.

(b) Find the points of inflection of $F(x)$ and determine whether the concavity changes from up to down or from down to up.

(c) GU Plot $f(x)$ and $F(x)$ on the same set of axes and confirm your answers to (a) and (b).

SOLUTION

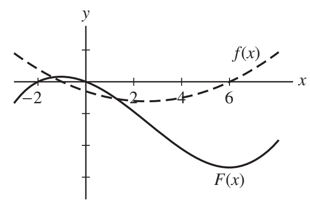
(a) If $F(x) = \int_0^x (t^2 - 5t - 6)\,dt$, then $F'(x) = x^2 - 5x - 6$ and $F''(x) = 2x - 5$. Solving $F'(x) = x^2 - 5x - 6 = 0$ yields critical points $x = -1$ and $x = 6$. Since $F''(-1) = -7 < 0$, there is a local maximum value of F at $x = -1$. Moreover, since $F''(6) = 7 > 0$, there is a local minimum value of F at $x = 6$.

(b) As noted in part (a),

$$F'(x) = x^2 - 5x - 6 \qquad \text{and} \qquad F''(x) = 2x - 5.$$

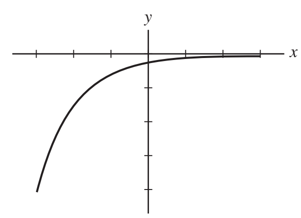
A candidate point of inflection occurs where $F''(x) = 2x - 5 = 0$. Thus $x = \frac{5}{2}$. $F''(x)$ changes from negative to positive at this point, so there is a point of inflection at $x = \frac{5}{2}$ and concavity changes from down to up.

(c) From the graph below, we clearly note that $F(x)$ has a local maximum at $x = -1$, a local minimum at $x = 6$ and a point of inflection at $x = \frac{5}{2}$.



47. Sketch the graph of an increasing function $f(x)$ such that both $f'(x)$ and $A(x) = \int_0^x f(t)\,dt$ are decreasing.

SOLUTION If $f'(x)$ is decreasing, then $f''(x)$ must be negative. Furthermore, if $A(x) = \int_0^x f(t)\,dt$ is decreasing, then $A'(x) = f(x)$ must also be negative. Thus, we need a function which is negative but increasing and concave down. The graph of one such function is shown below.



48. Figure 13 shows the graph of $f(x) = x \sin x$. Let $F(x) = \int_0^x t \sin t\,dt$.

(a) Locate the local max and absolute max of $F(x)$ on $[0, 3\pi]$.

(b) Justify graphically: $F(x)$ has precisely one zero in $[\pi, 2\pi]$.

(c) How many zeros does $F(x)$ have in $[0, 3\pi]$?

(d) Find the inflection points of $F(x)$ on $[0, 3\pi]$. For each one, state whether the concavity changes from up to down or from down to up.

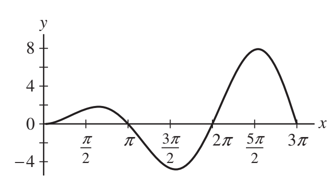


FIGURE 13 Graph of $f(x) = x \sin x$.

SOLUTION Let $F(x) = \int_0^x t \sin t\,dt$. A graph of $f(x) = x \sin x$ is depicted in Figure 13. Note that $F'(x) = f(x)$ and $F''(x) = f'(x)$.

(a) For F to have a local maximum at $x_0 \in (0, 3\pi)$ we must have $F'(x_0) = f(x_0) = 0$ and $F' = f$ must change sign from + to 0 to − as x increases through x_0. This occurs at $x = \pi$. The absolute maximum of $F(x)$ on $[0, 3\pi]$ occurs at $x = 3\pi$ since (from the figure) the signed area between $x = 0$ and $x = c$ is greatest for $x = c = 3\pi$.

(b) At $x = \pi$, the value of F is positive since $f(x) > 0$ on $(0, \pi)$. As x increases along the interval $[\pi, 2\pi]$, we see that F decreases as the negatively signed area accumulates. Eventually the additional negatively signed area "outweighs" the prior positively signed area and F attains the value 0, say at $b \in (\pi, 2\pi)$. Thereafter, on $(b, 2\pi)$, we see that f is negative and thus F becomes and continues to be negative as the negatively signed area accumulates. Therefore, $F(x)$ takes the value 0 exactly once in the interval $[\pi, 2\pi]$.

(c) $F(x)$ has two zeroes in $[0, 3\pi]$. One is described in part (b) and the other must occur in the interval $[2\pi, 3\pi]$ because $F(x) < 0$ at $x = 2\pi$ but clearly the positively signed area over $[2\pi, 3\pi]$ is greater than the previous negatively signed area.

(d) Since f is differentiable, we have that F is twice differentiable on I. Thus $F(x)$ has an inflection point at x_0 provided $F''(x_0) = f'(x_0) = 0$ *and* $F''(x) = f'(x)$ changes sign at x_0. If $F'' = f'$ changes sign from + to 0 to − at x_0, then f has a local maximum at x_0. There is clearly such a value x_0 in the figure in the interval $[\pi/2, \pi]$ and another around $5\pi/2$. Accordingly, F has two inflection points where $F(x)$ changes from concave up to concave down. If $F'' = f'$ changes sign from − to 0 to + at x_0, then f has a local minimum at x_0. From the figure, there is such an x_0 around $3\pi/2$; so F has one inflection point where $F(x)$ changes from concave down to concave up.

49. GU Find the smallest positive critical point of

$$F(x) = \int_0^x \cos(t^{3/2})\,dt$$

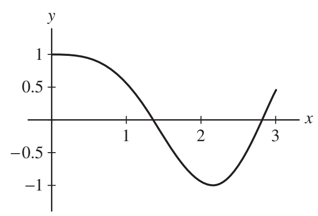
and determine whether it is a local min or max. Then find the smallest positive inflection point of $F(x)$ and use a graph of $y = \cos(x^{3/2})$ to determine whether the concavity changes from up to down or from down to up.

SOLUTION A critical point of $F(x)$ occurs where $F'(x) = \cos(x^{3/2}) = 0$. The smallest positive critical points occurs where $x^{3/2} = \pi/2$, so that $x = (\pi/2)^{2/3}$. $F'(x)$ goes from positive to negative at this point, so $x = (\pi/2)^{2/3}$ corresponds to a local maximum..

Candidate inflection points of $F(x)$ occur where $F''(x) = 0$. By FTC, $F'(x) = \cos(x^{3/2})$, so $F''(x) = -(3/2)x^{1/2}\sin(x^{3/2})$. Finding the smallest positive solution of $F''(x) = 0$, we get:

$$\begin{aligned} -(3/2)x^{1/2}\sin(x^{3/2}) &= 0 \\ \sin(x^{3/2}) &= 0 \qquad \text{(since } x > 0) \\ x^{3/2} &= \pi \\ x &= \pi^{2/3} \approx 2.14503. \end{aligned}$$

From the plot below, we see that $F'(x) = \cos(x^{3/2})$ changes from decreasing to increasing at $\pi^{2/3}$, so $F(x)$ changes from concave down to concave up at that point.



Further Insights and Challenges

50. Proof of FTC II The proof in the text assumes that $f(x)$ is increasing. To prove it for all continuous functions, let $m(h)$ and $M(h)$ denote the *minimum* and *maximum* of $f(t)$ on $[x, x+h]$ (Figure 14). The continuity of $f(x)$ implies that $\lim_{h\to 0} m(h) = \lim_{h\to 0} M(h) = f(x)$. Show that for $h > 0$,

$$hm(h) \le A(x+h) - A(x) \le hM(h)$$

For $h < 0$, the inequalities are reversed. Prove that $A'(x) = f(x)$.

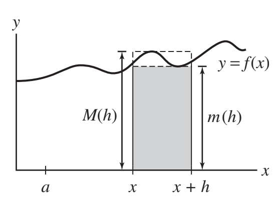


FIGURE 14 Graphical interpretation of $A(x+h) - A(x)$.

SOLUTION Let $f(x)$ be continuous on $[a, b]$. For $h > 0$, let $m(h)$ and $M(h)$ denote the minimum and maximum values of f on $[x, x+h]$. Since f is continuous, we have $\lim_{h\to 0+} m(h) = \lim_{h\to 0+} M(h) = f(x)$. If $h > 0$, then since $m(h) \le f(x) \le M(h)$ on $[x, x+h]$, we have

$$hm(h) = \int_x^{x+h} m(h)\,dt \le \int_x^{x+h} f(t)\,dt = A(x+h) - A(x) = \int_x^{x+h} f(t)\,dt \le \int_x^{x+h} M(h)\,dt = hM(h).$$

In other words, $hm(h) \le A(x+h) - A(x) \le hM(h)$. Since $h > 0$, it follows that $m(h) \le \dfrac{A(x+h)-A(x)}{h} \le M(h)$. Letting $h \to 0+$ yields

$$f(x) \le \lim_{h\to 0+} \frac{A(x+h)-A(x)}{h} \le f(x),$$

whence

$$\lim_{h\to 0+} \frac{A(x+h)-A(x)}{h} = f(x)$$

by the Squeeze Theorem. If $h < 0$, then

$$-hm(h) = \int_{x+h}^{x} m(h)\,dt \le \int_{x+h}^{x} f(t)\,dt = A(x) - A(x+h) = \int_{x+h}^{x} f(t)\,dt \le \int_{x+h}^{x} M(h)\,dt = -hM(h).$$

Since $h < 0$, we have $-h > 0$ and thus

$$m(h) \le \frac{A(x)-A(x+h)}{-h} \le M(h)$$

or

$$m(h) \le \frac{A(x+h)-A(x)}{h} \le M(h).$$

Letting $h \to 0-$ gives

$$f(x) \le \lim_{h\to 0-} \frac{A(x+h)-A(x)}{h} \le f(x),$$

so that

$$\lim_{h\to 0-} \frac{A(x+h)-A(x)}{h} = f(x)$$

by the Squeeze Theorem. Since the one-sided limits agree, we therefore have

$$A'(x) = \lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = f(x).$$

51. Proof of FTC I FTC I asserts that $\int_a^b f(t)\,dt = F(b) - F(a)$ if $F'(x) = f(x)$. Use FTC II to give a new proof of FTC I as follows. Set $A(x) = \int_a^x f(t)\,dt$.

(a) Show that $F(x) = A(x) + C$ for some constant.

(b) Show that $F(b) - F(a) = A(b) - A(a) = \displaystyle\int_a^b f(t)\,dt$.

SOLUTION Let $F'(x) = f(x)$ and $A(x) = \int_a^x f(t)\,dt$.

(a) Then by the FTC, Part II, $A'(x) = f(x)$ and thus $A(x)$ and $F(x)$ are both antiderivatives of $f(x)$. Hence $F(x) = A(x) + C$ for some constant C.

(b)

$$\begin{aligned} F(b) - F(a) &= (A(b) + C) - (A(a) + C) = A(b) - A(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = \int_a^b f(t)\,dt - 0 = \int_a^b f(t)\,dt \end{aligned}$$

which proves the FTC, Part I.

52. Can Every Antiderivative Be Expressed as an Integral? The area function $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$ for every value of a. However, not all antiderivatives are obtained in this way. The general antiderivative of $f(x) = x$ is $F(x) = \frac{1}{2}x^2 + C$. Show that $F(x)$ is an area function if $C \le 0$ but not if $C > 0$.

SOLUTION Let $f(x) = x$. The general antiderivative of $f(x)$ is $F(x) = \frac{1}{2}x^2 + C$. Let $A(x) = \int_a^x f(t)\,dt = \int_a^x t\,dt = \frac{1}{2}t^2\Big|_a^x = \frac{1}{2}x^2 - \frac{1}{2}a^2$ be an area function of $f(x) = x$. To express $F(x)$ as an area function, we must find a value for a such that $\frac{1}{2}x^2 - \frac{1}{2}a^2 = \frac{1}{2}x^2 + C$, whence $a = \pm\sqrt{-2C}$. If $C \le 0$, then $-2C \ge 0$ and we may choose either $a = \sqrt{-2C}$ or $a = -\sqrt{-2C}$. However, if $C > 0$, then there is no real solution for a and $F(x)$ cannot be expressed as an area function.

53. Prove the formula

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))v'(x) - f(u(x))u'(x)$$

SOLUTION Write

$$\int_{u(x)}^{v(x)} f(x)\,dx = \int_{u(x)}^{0} f(x)\,dx + \int_{0}^{v(x)} f(x)\,dx = \int_{0}^{v(x)} f(x)\,dx - \int_{0}^{u(x)} f(x)\,dx.$$

Then, by the Chain Rule and the FTC,

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x)\,dx = \frac{d}{dx}\int_{0}^{v(x)} f(x)\,dx - \frac{d}{dx}\int_{0}^{u(x)} f(x)\,dx$$
$$= f(v(x))v'(x) - f(u(x))u'(x).$$

54. Use the result of Exercise 53 to calculate

$$\frac{d}{dx}\int_{\ln x}^{e^x} \sin t\,dt$$

SOLUTION By Exercise 53,

$$\frac{d}{dx}\int_{\ln x}^{e^x} \sin t\,dt = e^x \sin e^x - \frac{1}{x}\sin \ln x.$$

5.5 Net Change as the Integral of a Rate

Preliminary Questions

1. A hot metal object is submerged in cold water. The rate at which the object cools (in degrees per minute) is a function $f(t)$ of time. Which quantity is represented by the integral $\int_0^T f(t)\,dt$?

SOLUTION The definite integral $\int_0^T f(t)\,dt$ represents the total drop in temperature of the metal object in the first T minutes after being submerged in the cold water.

2. A plane travels 560 km from Los Angeles to San Francisco in 1 hour. If the plane's velocity at time t is $v(t)$ km/h, what is the value of $\int_0^1 v(t)\,dt$?

SOLUTION The definite integral $\int_0^1 v(t)\,dt$ represents the total distance traveled by the airplane during the one hour flight from Los Angeles to San Francisco. Therefore the value of $\int_0^1 v(t)\,dt$ is 560 km.

3. Which of the following quantities would be naturally represented as derivatives and which as integrals?

(a) Velocity of a train
(b) Rainfall during a 6-month period
(c) Mileage per gallon of an automobile
(d) Increase in the U.S. population from 1990 to 2010

SOLUTION Quantities **(a)** and **(c)** involve rates of change, so these would naturally be represented as derivatives. Quantities **(b)** and **(d)** involve an accumulation, so these would naturally be represented as integrals.

Exercises

1. Water flows into an empty reservoir at a rate of $3000 + 20t$ liters per hour. What is the quantity of water in the reservoir after 5 hours?

SOLUTION The quantity of water in the reservoir after five hours is

$$\int_0^5 (3000 + 20t)\,dt = \left(3000t + 10t^2\right)\Big|_0^5 = 15{,}250 \text{ gallons}.$$

2. A population of insects increases at a rate of $200 + 10t + 0.25t^2$ insects per day. Find the insect population after 3 days, assuming that there are 35 insects at $t = 0$.

SOLUTION The increase in the insect population over three days is

$$\int_0^3 \left(200 + 10t + \frac{1}{4}t^2\right)dt = \left(200t + 5t^2 + \frac{1}{12}t^3\right)\Big|_0^3 = \frac{2589}{4} = 647.25.$$

Accordingly, the population after 3 days is $35 + 647.25 = 682.25$ or 682 insects.

3. A survey shows that a mayoral candidate is gaining votes at a rate of $2000t + 1000$ votes per day, where t is the number of days since she announced her candidacy. How many supporters will the candidate have after 60 days, assuming that she had no supporters at $t = 0$?

SOLUTION The number of supporters the candidate has after 60 days is

$$\int_0^{60} (2000t + 1000)\, dt = (1000t^2 + 1000t)\Big|_0^{60} = 3{,}660{,}000.$$

4. A factory produces bicycles at a rate of $95 + 3t^2 - t$ bicycles per week. How many bicycles were produced from the beginning of week 2 to the end of week 3?

SOLUTION The rate of production is $r(t) = 95 + 3t^2 - t$ bicycles per week and the period from the beginning of week 2 to the end of week 3 corresponds to the second and third weeks of production. Accordingly, the number of bikes produced from the beginning of week 2 to the end of week 3 is

$$\int_1^3 r(t)\, dt = \int_1^3 \left(95 + 3t^2 - t\right)\, dt = \left(95t + t^3 - \frac{1}{2}t^2\right)\Big|_1^3 = 212$$

bicycles.

5. Find the displacement of a particle moving in a straight line with velocity $v(t) = 4t - 3$ m/s over the time interval $[2, 5]$.

SOLUTION The displacement is given by

$$\int_2^5 (4t - 3)\, dt = (2t^2 - 3t)\Big|_2^5 = (50 - 15) - (8 - 6) = 33\text{m}.$$

6. Find the displacement over the time interval $[1, 6]$ of a helicopter whose (vertical) velocity at time t is $v(t) = 0.02t^2 + t$ m/s.

SOLUTION Given $v(t) = \frac{1}{50}t^2 + t$ m/s, the change in height over $[1, 6]$ is

$$\int_1^6 v(t)\, dt = \int_1^6 \left(\frac{1}{50}t^2 + t\right)\, dt = \left(\frac{1}{150}t^3 + \frac{1}{2}t^2\right)\Big|_1^6 = \frac{284}{15} \approx 18.93 \text{ m}.$$

7. A cat falls from a tree (with zero initial velocity) at time $t = 0$. How far does the cat fall between $t = 0.5$ and $t = 1$ s? Use Galileo's formula $v(t) = -9.8t$ m/s.

SOLUTION Given $v(t) = -9.8t$ m/s, the total distance the cat falls during the interval $[\frac{1}{2}, 1]$ is

$$\int_{1/2}^1 |v(t)|\, dt = \int_{1/2}^1 9.8t\, dt = 4.9t^2\Big|_{1/2}^1 = 4.9 - 1.225 = 3.675 \text{ m}.$$

8. A projectile is released with an initial (vertical) velocity of 100 m/s. Use the formula $v(t) = 100 - 9.8t$ for velocity to determine the distance traveled during the first 15 seconds.

SOLUTION The distance traveled is given by

$$\int_0^{15} |100 - 9.8t|\, dt = \int_0^{100/9.8} (100 - 9.8t)\, dt + \int_{100/9.8}^{15} (9.8t - 100)\, dt$$
$$= \left(100t - 4.9t^2\right)\Big|_0^{100/9.8} + \left(4.9t^2 - 100t\right)\Big|_{100/9.8}^{15} \approx 622.9 \text{ m}.$$

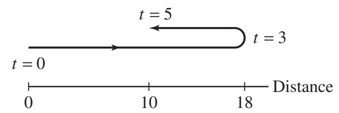
In Exercises 9–12, a particle moves in a straight line with the given velocity (in m/s). Find the displacement and distance traveled over the time interval, and draw a motion diagram like Figure 3 (with distance and time labels).

9. $v(t) = 12 - 4t, \quad [0, 5]$

SOLUTION Displacement is given by $\int_0^5 (12 - 4t)\, dt = (12t - 2t^2)\Big|_0^5 = 10$ ft, while total distance is given by

$$\int_0^5 |12 - 4t|\, dt = \int_0^3 (12 - 4t)\, dt + \int_3^5 (4t - 12)\, dt = (12t - 2t^2)\Big|_0^3 + (2t^2 - 12t)\Big|_3^5 = 26 \text{ ft}.$$

The displacement diagram is given here.



10. $v(t) = 36 - 24t + 3t^2, \quad [0, 10]$

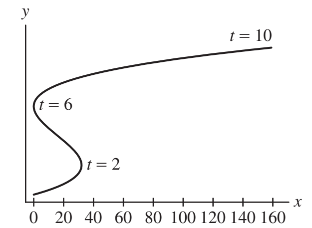
SOLUTION Let $v(t) = 36 - 24t + 3t^2 = 3(t-2)(t-6)$. Displacement is given by

$$\int_0^{10} (36 - 24t + 3t^2)\,dt = \left(36t - 12t^2 + t^3\right)\Big|_0^{10} = 160$$

meters. Total distance traveled is given by

$$\begin{aligned}\int_0^{10} |36 - 24t + 3t^2|\,dt &= \int_0^{2} (36 - 24t + 3t^2)\,dt + \int_2^{6} (24t - 36 - 3t^2)\,dt + \int_6^{10} (36 - 24t + 3t^2)\,dt \\ &= \left(36t - 12t^2 + t^3\right)\Big|_0^{2} + \left(12t^2 - 36t - t^3\right)\Big|_0^{10} + \left(36t - 12t^2 + t^3\right)\Big|_6^{10} \\ &= 224 \text{ meters.}\end{aligned}$$

The displacement diagram is given here.

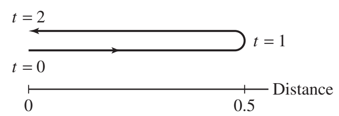


11. $v(t) = t^{-2} - 1, \quad [0.5, 2]$

SOLUTION Displacement is given by $\int_{0.5}^{2} (t^{-2} - 1)\,dt = \left(-t^{-1} - t\right)\Big|_{0.5}^{2} = 0$ m, while total distance is given by

$$\int_{0.5}^{2} \left|t^{-2} - 1\right|\,dt = \int_{0.5}^{1} (t^{-2} - 1)\,dt + \int_1^{2} (1 - t^{-2})\,dt = \left(-t^{-1} - t\right)\Big|_{0.5}^{1} + \left(t + t^{-1}\right)\Big|_1^{2} = 1 \text{ m.}$$

The displacement diagram is given here.

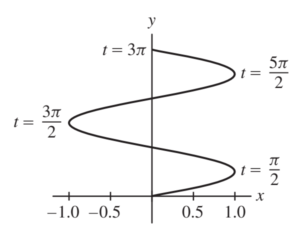


12. $v(t) = \cos t, \quad [0, 3\pi]$

SOLUTION Displacement is given by $\int_0^{3\pi} \cos t\,dt = \sin t\Big|_0^{3\pi} = 0$ meters, while the total distance traveled is given by

$$\begin{aligned}\int_0^{3\pi} |\cos t|\,dt &= \int_0^{\pi/2} \cos t\,dt - \int_{\pi/2}^{3\pi/2} \cos t\,dt + \int_{3\pi/2}^{5\pi/2} \cos t\,dt - \int_{5\pi/2}^{3\pi} \cos t\ , dt \\ &= \sin t\Big|_0^{\pi/2} - \sin t\Big|_{\pi/2}^{3\pi/2} + \sin t\Big|_{3\pi/2}^{5\pi/2} - \sin t\Big|_{5\pi/2}^{3\pi} \\ &= 6 \text{ meters.}\end{aligned}$$

The displacement diagram is given here.



13. Find the net change in velocity over [1, 4] of an object with $a(t) = 8t - t^2$ m/s^2.

SOLUTION The net change in velocity is

$$\int_1^4 a(t)\,dt = \int_1^4 (8t - t^2)\,dt = \left(4t^2 - \frac{1}{3}t^3\right)\Big|_1^4 = 39 \text{ m/s}.$$

14. Show that if acceleration is constant, then the change in velocity is proportional to the length of the time interval.

SOLUTION Let $a(t) = a$ be the constant acceleration. Let $v(t)$ be the velocity. Let $[t_1, t_2]$ be the time interval concerned. We know that $v'(t) = a$, and, by FTC,

$$v(t_2) - v(t_1) = \int_{t_1}^{t_2} a\,dt = a(t_2 - t_1),$$

So the net change in velocity is proportional to the length of the time interval with constant of proportionality a.

15. The traffic flow rate past a certain point on a highway is $q(t) = 3000 + 2000t - 300t^2$ (t in hours), where $t = 0$ is 8 AM. How many cars pass by in the time interval from 8 to 10 AM?

SOLUTION The number of cars is given by

$$\int_0^2 q(t)\,dt = \int_0^2 (3000 + 2000t - 300t^2)\,dt = \left(3000t + 1000t^2 - 100t^3\right)\Big|_0^2$$
$$= 3000(2) + 1000(4) - 100(8) = 9200 \text{ cars}.$$

16. The marginal cost of producing x tablet computers is $C'(x) = 120 - 0.06x + 0.00001x^2$ What is the cost of producing 3000 units if the setup cost is \$90,000? If production is set at 3000 units, what is the cost of producing 200 additional units?

SOLUTION The production coot for producing 3000 units is

$$\int_0^{3000} (120 - 0.06x + 0.00001x^2)\,dx = \left(120x - 0.03x^2 + \frac{1}{3}0.00001x^3\right)\Big|_0^{3000}$$
$$= 360{,}000 - 270{,}000 + 90{,}000 = 180{,}000$$

dollars. Adding in the setup cost, we find the total cost of producing 3000 units is \$270,000. If production is set at 3000 units, the cost of producing an additional 200 units is

$$\int_{3000}^{3200} (120 - 0.06x + 0.00001x^2)\,dx = \left(120x - 0.03x^2 + \frac{1}{3}0.00001x^3\right)\Big|_{3000}^{3200}$$
$$= 384{,}000 - 307{,}200 + 109{,}226.67 - 180{,}000$$

or \$6026.67.

17. A small boutique produces wool sweaters at a marginal cost of $40 - 5[[x/5]]$ for $0 \le x \le 20$, where $[[x]]$ is the greatest integer function. Find the cost of producing 20 sweaters. Then compute the average cost of the first 10 sweaters and the last 10 sweaters.

SOLUTION The total cost of producing 20 sweaters is

$$\int_0^{20} (40 - 5[[x/5]])\,dx = \int_0^5 40\,dx + \int_5^{10} 35\,dx + \int_{10}^{15} 30\,dx + \int_{15}^{20} 25\,dx$$
$$= 40(5) + 35(5) + 30(5) + 25(5) = 650 \text{ dollars}.$$

From this calculation, we see that the cost of the first 10 sweaters is \$375 and the cost of the last ten sweaters is \$275; thus, the average cost of the first ten sweaters is \$37.50 and the average cost of the last ten sweaters is \$27.50.

18. The rate (in liters per minute) at which water drains from a tank is recorded at half-minute intervals. Compute the average of the left- and right-endpoint approximations to estimate the total amount of water drained during the first 3 minutes.

t (min)	0	0.5	1	1.5	2	2.5	3
r (l/min)	50	48	46	44	42	40	38

SOLUTION Let $\Delta t = 0.5$. Then

$$R_N = 0.5(48 + 46 + 44 + 42 + 40 + 38) = 129.0 \text{ liters}$$

$$L_N = 0.5(50 + 48 + 46 + 44 + 42 + 40) = 135.0 \text{ liters}$$

The average of R_N and L_N is $\frac{1}{2}(129 + 135) = 132$ liters.

19. The velocity of a car is recorded at half-second intervals (in feet per second). Use the average of the left- and right-endpoint approximations to estimate the total distance traveled during the first 4 seconds.

t	0	0.5	1	1.5	2	2.5	3	3.5	4
$v(t)$	0	12	20	29	38	44	32	35	30

SOLUTION Let $\Delta t = .5$. Then

$$R_N = 0.5 \cdot (12 + 20 + 29 + 38 + 44 + 32 + 35 + 30) = 120 \text{ ft.}$$

$$L_N = 0.5 \cdot (0 + 12 + 20 + 29 + 38 + 44 + 32 + 35) = 105 \text{ ft.}$$

The average of R_N and L_N is 112.5 ft.

20. To model the effects of a **carbon tax** on CO_2 emissions, policymakers study the *marginal cost of abatement* $B(x)$, defined as the cost of increasing CO_2 reduction from x to $x + 1$ tons (in units of ten thousand tons—Figure 4). Which quantity is represented by the area under the curve over [0, 3] in Figure 4?

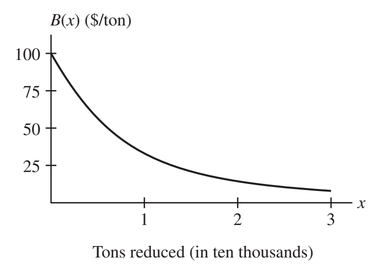


FIGURE 4 Marginal cost of abatement $B(x)$.

SOLUTION The area under the curve over [0, 3] represents the total cost of reducing the amount of CO_2 released into the atmosphere by 3 tons.

21. A megawatt of power is 10^6 W, or 3.6×10^9 J/hour. Which quantity is represented by the area under the graph in Figure 5? Estimate the energy (in joules) consumed during the period 4 PM to 8 PM.

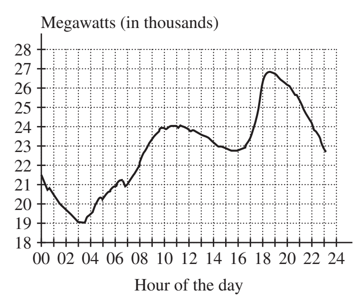


FIGURE 5 Power consumption over 1-day period in California (February 2010).

SOLUTION The area under the graph in Figure 5 represents the total power consumption over one day in California. Assuming $t = 0$ corresponds to midnight, the period 4 PM to 8 PM corresponds to $t = 16$ to $t = 20$. The left and right endpoint approximations are

$$L = 1(22.8 + 23.5 + 26.1 + 26.7) = 99.1 \text{megawatt} \cdot \text{hours}$$

$$R = 1(23.5 + 26.1 + 26.7 + 26.1) = 102.4 \text{megawatt} \cdot \text{hours}$$

The average of these values is

$$100.75 \text{megawatt} \cdot \text{hours} = 3.627 \times 10^{11} \text{ joules.}$$

22. Figure 6 shows the migration rate $M(t)$ of Ireland in the period 1988–1998. This is the rate at which people (in thousands per year) move into or out of the country.

(a) Is the following integral positive or negative? What does this quantity represent?

$$\int_{1988}^{1998} M(t)\,dt$$

(b) Did migration in the period 1988–1998 result in a net influx of people into Ireland or a net outflow of people from Ireland?

(c) During which two years could the Irish prime minister announce, "We've hit an inflection point. We are still losing population, but the trend is now improving."

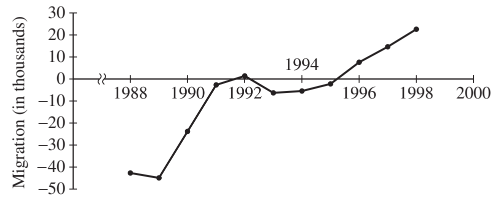


FIGURE 6 Irish migration rate (in thousands per year).

SOLUTION

(a) Because there appears to be more area below the t-axis than above in Figure 6,

$$\int_{1988}^{1998} M(t)\,dt$$

is negative. This quantity represents the net migration from Ireland during the period 1988–1998.

(b) As noted in part (a), there appears to be more area below the t-axis than above in Figure 6, so migration in the period 1988–1998 resulted in a net outflow of people from Ireland.

(c) The prime minister can make this statement when the graph of M is at a local minimum, which appears to be in the years 1989 and 1993.

23. Let $N(d)$ be the number of asteroids of diameter $\le d$ kilometers. Data suggest that the diameters are distributed according to a piecewise power law:

$$N'(d) = \begin{cases} 1.9 \times 10^9 d^{-2.3} & \text{for } d < 70 \\ 2.6 \times 10^{12} d^{-4} & \text{for } d \ge 70 \end{cases}$$

(a) Compute the number of asteroids with diameter between 0.1 and 100 km.

(b) Using the approximation $N(d+1) - N(d) \approx N'(d)$, estimate the number of asteroids of diameter 50 km.

SOLUTION

(a) The number of asteroids with diameter between 0.1 and 100 km is

$$\begin{aligned}\int_{0.1}^{100} N'(d)\,dd &= \int_{0.1}^{70} 1.9 \times 10^9 d^{-2.3}\,dd + \int_{70}^{100} 2.6 \times 10^{12} d^{-4}\,dd \\ &= -\frac{1.9 \times 10^9}{1.3} d^{-1.3}\Big|_{0.1}^{70} - \frac{2.6 \times 10^{12}}{3} d^{-3}\Big|_{70}^{100} \\ &= 2.916 \times 10^{10} + 1.66 \times 10^6 \approx 2.916 \times 10^{10}.\end{aligned}$$

(b) Taking $d = 49.5$,

$$N(50.5) - N(49.5) \approx N'(49.5) = 1.9 \times 10^9 49.5^{-2.3} = 240{,}525.79.$$

Thus, there are approximately 240,526 asteroids of diameter 50 km.

24. Heat Capacity The heat capacity $C(T)$ of a substance is the amount of energy (in joules) required to raise the temperature of 1 g by 1°C at temperature T.

(a) Explain why the energy required to raise the temperature from T_1 to T_2 is the area under the graph of $C(T)$ over $[T_1, T_2]$.

(b) How much energy is required to raise the temperature from 50 to 100°C if $C(T) = 6 + 0.2\sqrt{T}$?

SOLUTION

(a) Since $C(T)$ is the energy required to raise the temperature of one gram of a substance by one degree when its temperature is T, the total energy required to raise the temperature from T_1 to T_2 is given by the definite integral $\int_{T_1}^{T_2} C(T)\,dT$. As $C(T) > 0$, the definite integral also represents the area under the graph of $C(T)$.

(b) If $C(T) = 6 + .2\sqrt{T} = 6 + \frac{1}{5}T^{1/2}$, then the energy required to raise the temperature from 50°C to 100°C is $\int_{50}^{100} C(T)\,dT$ or

$$\int_{50}^{100}\left(6+\frac{1}{5}T^{1/2}\right)dT = \left(6T + \frac{2}{15}T^{3/2}\right)\Big|_{50}^{100} = \left(6(100) + \frac{2}{15}(100)^{3/2}\right) - \left(6(50) + \frac{2}{15}(50)^{3/2}\right)$$

$$= \frac{1300 - 100\sqrt{2}}{3} \approx 386.19 \text{ Joules}$$

25. Figure 7 shows the rate $R(t)$ of natural gas consumption (in billions of cubic feet per day) in the mid-Atlantic states (New York, New Jersey, Pennsylvania). Express the total quantity of natural gas consumed in 2009 as an integral (with respect to time t in days). Then estimate this quantity, given the following monthly values of $R(t)$:

3.18, 2.86, 2.39, 1.49, 1.08, 0.80,
1.01, 0.89, 0.89, 1.20, 1.64, 2.52

Keep in mind that the number of days in a month varies with the month.

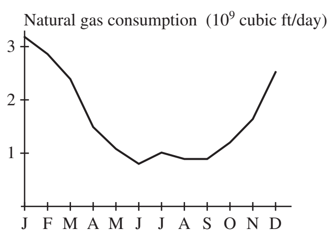


FIGURE 7 Natural gas consumption in 2009 in the mid-Atlantic states

SOLUTION The total quantity of natural gas consumed is given by

$$\int_0^{365} R(t)\,dt.$$

With the given data, we find

$$\int_0^{365} R(t)\,dt \approx 31(3.18) + 28(2.86) + 31(2.39) + 30(1.49) + 31(1.08) + 30(0.80)$$

$$+31(1.01) + 31(0.89) + 30(0.89) + 31(1.20) + 30(1.64) + 31(2.52)$$

$$= 605.05 \text{ billion cubic feet.}$$

26. Cardiac output is the rate R of volume of blood pumped by the heart per unit time (in liters per minute). Doctors measure R by injecting A mg of dye into a vein leading into the heart at $t = 0$ and recording the concentration $c(t)$ of dye (in milligrams per liter) pumped out at short regular time intervals (Figure 8).

(a) Explain: The quantity of dye pumped out in a small time interval $[t, t + \Delta t]$ is approximately $Rc(t)\Delta t$.

(b) Show that $A = R\int_0^T c(t)\,dt$, where T is large enough that all of the dye is pumped through the heart but not so large that the dye returns by recirculation.

(c) Assume $A = 5$ mg. Estimate R using the following values of $c(t)$ recorded at 1-second intervals from $t = 0$ to $t = 10$:

0, 0.4, 2.8, 6.5, 9.8, 8.9,
6.1, 4, 2.3, 1.1, 0

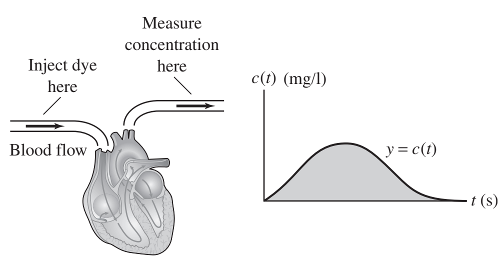


FIGURE 8

SOLUTION

(a) Over a short time interval, $c(t)$ is nearly constant. $Rc(t)$ is the rate of volume of dye (amount of fluid $\times$ concentration of dye in fluid) flowing out of the heart (in mg per minute). Over the short time interval $[t, t + \Delta t]$, the rate of flow of dye is approximately constant at $Rc(t)$ mg/minute. Therefore, the flow of dye over the interval is approximately $Rc(t)\Delta t$ mg.

(b) The rate of flow of dye is $Rc(t)$. Therefore the net flow between time $t = 0$ and time $t = T$ is

$$\int_0^T Rc(t)\, dt = R \int_0^T c(t)\, dt.$$

If T is great enough that all of the dye is pumped through the heart, the net flow is equal to all of the dye, so

$$A = R \int_0^T c(t)\, dt.$$

(c) In the table, $\Delta t = \frac{1}{60}$ minute, and $N = 10$. The right and left hand approximations of $\int_0^T c(t)\, dt$ are:

$$R_{10} = \frac{1}{60}(0.4 + 2.8 + 6.5 + 9.8 + 8.9 + 6.1 + 4 + 2.3 + 1.1 + 0) = 0.6983\ \frac{\text{mg} \cdot \text{minute}}{\text{liter}}$$
$$L_{10} = \frac{1}{60}(0 + 0.4 + 2.8 + 6.5 + 9.8 + 8.9 + 6.1 + 4 + 2.3 + 1.1) = 0.6983\ \frac{\text{mg} \cdot \text{minute}}{\text{liter}}$$

Both L_N and R_N are the same, so the average of L_N and R_N is 0.6983. Hence,

$$A = R \int_0^T c(t) dt$$
$$5\text{ mg} = R\left(0.6983\ \frac{\text{mg} \cdot \text{minute}}{\text{liter}}\right)$$
$$R = \frac{5}{0.6983}\ \frac{\text{liters}}{\text{minute}} = 7.16\ \frac{\text{liters}}{\text{minute}}.$$

Exercises 27 and 28: A study suggests that the extinction rate $r(t)$ of marine animal families during the Phanerozoic Eon can be modeled by the function $r(t) = 3130/(t + 262)$ for $0 \le t \le 544$, where t is time elapsed (in millions of years) since the beginning of the eon 544 million years ago. Thus, $t = 544$ refers to the present time, $t = 540$ is 4 million years ago, and so on.

27. Compute the average of R_N and L_N with $N = 5$ to estimate the total number of families that became extinct in the periods $100 \le t \le 150$ and $350 \le t \le 400$.

SOLUTION

- $(100 \le t \le 150)$ For $N = 5$,

$$\Delta t = \frac{150 - 100}{5} = 10.$$

  The table of values $\{r(t_i)\}_{i=0\ldots5}$ is given below:

t_i	100	110	120	130	140	150
$r(t_i)$	8.64641	8.41398	8.19372	7.98469	7.78607	7.59709

  The endpoint approximations are:

$$R_N = 10(8.41398 + 8.19372 + 7.98469 + 7.78607 + 7.59709) \approx 399.756 \text{ families}$$
$$L_N = 10(8.64641 + 8.41398 + 8.19372 + 7.98469 + 7.78607) \approx 410.249 \text{ families}$$

  The right endpoint approximation estimates 399.756 families became extinct in the period $100 \le t \le 150$, the left endpoint approximation estimates 410.249 families became extinct during this time. The average of the two is 405.362 families.

- $(350 \le t \le 400)$ For $N = 10$,

$$\Delta t = \frac{400 - 350}{5} = 19.$$

The table of values $\{r(t_i)\}_{i=0\ldots5}$ is given below:

t_i	350	360	370	380	390	400
$r(t_i)$	5.11438	5.03215	4.95253	4.87539	4.80061	4.72810

The endpoint approximations are:

$$R_N = 10(5.03215 + 4.95253 + 4.87539 + 4.80061 + 4.72810) \approx 243.888 \text{ families}$$

$$L_N = 10(5.11438 + 5.03215 + 4.95253 + 4.87539 + 4.80061) \approx 247.751 \text{ families}$$

The right endpoint approximation estimates 243.888 families became extinct in the period $350 \le t \le 400$, the left endpoint approximation estimates 247.751 families became extinct during this time. The average of the two is 245.820 families.

28. CAS Estimate the total number of extinct families from $t = 0$ to the present, using M_N with $N = 544$.

SOLUTION We are estimating

$$\int_0^{544} \frac{3130}{(t+262)}\,dt$$

using M_N with $N = 544$. If $N = 544$, $\Delta t = \dfrac{544-0}{544} = 1$ and $\{t_i^*\}_{i=1,\ldots N} = i\Delta t - (\Delta t/2) = i - \frac{1}{2}$.

$$M_N = \Delta t \sum_{i=1}^{N} r(t_i^*) = 1 \cdot \sum_{i=1}^{544} \frac{3130}{261.5+i} = 3517.3021.$$

Thus, we estimate that 3517 families have become extinct over the past 544 million years.

Further Insights and Challenges

29. Show that a particle, located at the origin at $t = 1$ and moving along the x-axis with velocity $v(t) = t^{-2}$, will never pass the point $x = 2$.

SOLUTION The particle's velocity is $v(t) = s'(t) = t^{-2}$, an antiderivative for which is $F(t) = -t^{-1}$. Hence, the particle's position at time t is

$$s(t) = \int_1^t s'(u)\,du = F(u)\Big|_1^t = F(t) - F(1) = 1 - \frac{1}{t} < 1$$

for all $t \ge 1$. Thus, the particle will never pass $x = 1$, which implies it will never pass $x = 2$ either.

30. Show that a particle, located at the origin at $t = 1$ and moving along the x-axis with velocity $v(t) = t^{-1/2}$ moves arbitrarily far from the origin after sufficient time has elapsed.

SOLUTION The particle's velocity is $v(t) = s'(t) = t^{-1/2}$, an antiderivative for which is $F(t) = 2t^{1/2}$. Hence, the particle's position at time t is

$$s(t) = \int_1^t s'(u)\,du = F(u)\Big|_1^t = F(t) - F(1) = 2\sqrt{t} - 1$$

for all $t \ge 1$. Let $S > 0$ denote an arbitrarily large distance from the origin. We see that for

$$t > \left(\frac{S+1}{2}\right)^2,$$

the particle will be more than S units from the origin. In other words, the particle moves arbitrarily far from the origin after sufficient time has elapsed.

5.6 Substitution Method

Preliminary Questions

1. Which of the following integrals is a candidate for the Substitution Method?

(a) $\int 5x^4 \sin(x^5)\,dx$ **(b)** $\int \sin^5 x\, \cos x\,dx$ **(c)** $\int x^5 \sin x\,dx$

SOLUTION The function in **(c)**: $x^5 \sin x$ is not of the form $g(u(x))u'(x)$. The function in **(a)** meets the prescribed pattern with $g(u) = \sin u$ and $u(x) = x^5$. Similarly, the function in **(b)** meets the prescribed pattern with $g(u) = u^5$ and $u(x) = \sin x$.

2. Find an appropriate choice of u for evaluating the following integrals by substitution:

(a) $\int x(x^2+9)^4\,dx$ **(b)** $\int x^2 \sin(x^3)\,dx$ **(c)** $\int \sin x\, \cos^2 x\,dx$

SOLUTION

(a) $x(x^2+9)^4 = \frac{1}{2}(2x)(x^2+9)^4$; hence, $c = \frac{1}{2}$, $f(u) = u^4$, and $u(x) = x^2+9$.

(b) $x^2\sin(x^3) = \frac{1}{3}(3x^2)\sin(x^3)$; hence, $c = \frac{1}{3}$, $f(u) = \sin u$, and $u(x) = x^3$.

(c) $\sin x\cos^2 x = -(-\sin x)\cos^2 x$; hence, $c = -1$, $f(u) = u^2$, and $u(x) = \cos x$.

3. Which of the following is equal to $\int_0^2 x^2(x^3+1)\,dx$ for a suitable substitution?

(a) $\frac{1}{3}\int_0^2 u\,du$ **(b)** $\int_0^9 u\,du$ **(c)** $\frac{1}{3}\int_1^9 u\,du$

SOLUTION With the substitution $u = x^3+1$, the definite integral $\int_0^2 x^2(x^3+1)\,dx$ becomes $\frac{1}{3}\int_1^9 u\,du$. The correct answer is **(c)**.

Exercises

In Exercises 1–6, calculate du.

1. $u = x^3 - x^2$

SOLUTION Let $u = x^3 - x^2$. Then $du = (3x^2 - 2x)\,dx$.

2. $u = 2x^4 + 8x^{-1}$

SOLUTION Let $u = 2x^4 + 8x^{-1}$. Then $du = (8x^3 - 8x^{-2})\,dx$.

3. $u = \cos(x^2)$

SOLUTION Let $u = \cos(x^2)$. Then $du = -\sin(x^2)\cdot 2x\,dx = -2x\sin(x^2)\,dx$.

4. $u = \tan x$

SOLUTION Let $u = \tan x$. Then $du = \sec^2 x\,dx$.

5. $u = e^{4x+1}$

SOLUTION Let $u = e^{4x+1}$. Then $du = 4e^{4x+1}\,dx$.

6. $u = \ln(x^4+1)$

SOLUTION Let $u = \ln(x^4+1)$. Then $du = \dfrac{4x^3}{x^4+1}\,dx$.

In Exercises 7–22, write the integral in terms of u and du. Then evaluate.

7. $\int (x-7)^3\,dx, \quad u = x-7$

SOLUTION Let $u = x-7$. Then $du = dx$. Hence

$$\int (x-7)^3\,dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}(x-7)^4 + C.$$

8. $\int (x+25)^{-2}\,dx, \quad u = x+25$

SOLUTION Let $u = x+25$. Then $du = dx$ and

$$\int (x+25)^{-2}\,dx = \int u^{-2}\,du = -u^{-1} + C = -\frac{1}{x+25} + C.$$

9. $\int t\sqrt{t^2+1}\,dt, \quad u = t^2+1$

SOLUTION Let $u = t^2+1$. Then $du = 2t\,dt$. Hence,

$$\int t\sqrt{t^2+1}\,dt = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(t^2+1)^{3/2} + C.$$

10. $\int (x^3+1)\cos(x^4+4x)\,dx, \quad u = x^4+4x$

SOLUTION Let $u = x^4+4x$. Then $du = (4x^3+4)\,dx = 4(x^3+1)\,dx$ and

$$\int (x^3+1)\cos(x^4+4x)\,dx = \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+4x) + C.$$

11. $\int \frac{t^3}{(4-2t^4)^{11}}\,dt, \quad u = 4-2t^4$

SOLUTION Let $u = 4-2t^4$. Then $du = -8t^3\,dt$. Hence,

$$\int \frac{t^3}{(4-2t^4)^{11}}\,dt = -\frac{1}{8}\int u^{-11}\,du = \frac{1}{80}u^{-10} + C = \frac{1}{80}(4-2t^4)^{-10} + C.$$

12. $\int \sqrt{4x-1}\,dx, \quad u = 4x-1$

SOLUTION Let $u = 4x-1$. Then $du = 4\,dx$ or $\frac{1}{4}du = dx$. Hence

$$\int \sqrt{4u-1}\,dx = \frac{1}{4}\int u^{1/2}\,du = \frac{1}{4}\left(\frac{2}{3}u^{3/2}\right) + C = \frac{1}{6}(4x-1)^{3/2} + C.$$

13. $\int x(x+1)^9\,dx, \quad u = x+1$

SOLUTION Let $u = x+1$. Then $x = u-1$ and $du = dx$. Hence

$$\int x(x+1)^9\,dx = \int (u-1)u^9\,du = \int (u^{10}-u^9)\,du$$
$$= \frac{1}{11}u^{11} - \frac{1}{10}u^{10} + C = \frac{1}{11}(x+1)^{11} - \frac{1}{10}(x+1)^{10} + C.$$

14. $\int x\sqrt{4x-1}\,dx, \quad u = 4x-1$

SOLUTION Let $u = 4x-1$. Then $x = \frac{1}{4}(u+1)$ and $du = 4\,dx$ or $\frac{1}{4}\,du = dx$. Hence,

$$\int x\sqrt{4x-1}\,dx = \frac{1}{16}\int (u+1)u^{1/2}\,du = \frac{1}{16}\int (u^{3/2}+u^{1/2})\,du$$
$$= \frac{1}{16}\left(\frac{2}{5}u^{5/2}\right) + \frac{1}{16}\left(\frac{2}{3}u^{3/2}\right) + C$$
$$= \frac{1}{40}(4x-1)^{5/2} + \frac{1}{24}(4x-1)^{3/2} + C.$$

15. $\int x^2\sqrt{x+1}\,dx, \quad u = x+1$

SOLUTION Let $u = x+1$. Then $x = u-1$ and $du = dx$. Hence

$$\int x^2\sqrt{x+1}\,dx = \int (u-1)^2u^{1/2}\,du = \int (u^{5/2}-2u^{3/2}+u^{1/2})\,du$$
$$= \frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$$
$$= \frac{2}{7}(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2} + C.$$

16. $\int \sin(4\theta-7)\,d\theta, \quad u = 4\theta-7$

SOLUTION Let $u = 4\theta-7$. Then $du = 4\,d\theta$ and

$$\int \sin(4\theta-7)\,d\theta = \frac{1}{4}\int \sin u\,du = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos(4\theta-7) + C.$$

17. $\int \sin^2\theta\cos\theta\,d\theta, \quad u=\sin\theta$

SOLUTION Let $u=\sin\theta$. Then $du=\cos\theta\,d\theta$. Hence,

$$\int \sin^2\theta\cos\theta\,d\theta=\int u^2\,du=\frac{1}{3}u^3+C=\frac{1}{3}\sin^3\theta+C.$$

18. $\int \sec^2 x\tan x\,dx, \quad u=\tan x$

SOLUTION Let $u=\tan x$. Then $du=\sec^2 x\,dx$. Hence

$$\int \sec^2 x\tan x\,dx=\int u\,du=\frac{1}{2}u^2+C=\frac{1}{2}\tan^2 x+C.$$

19. $\int xe^{-x^2}\,dx, \quad u=-x^2$

SOLUTION Let $u=-x^2$. Then $du=-2x\,dx$ or $-\frac{1}{2}\,du=x\,dx$. Hence,

$$\int xe^{-x^2}\,dx=-\frac{1}{2}\int e^u\,du=-\frac{1}{2}e^u+C=-\frac{1}{2}e^{-x^2}+C.$$

20. $\int (\sec^2 t)e^{\tan t}\,dt, \quad u=\tan t$

SOLUTION Let $u=\tan t$. Then $du=\sec^2 t\,dt$ and

$$\int (\sec^2 t)e^{\tan t}\,dt=\int e^u\,du=e^u+C=e^{\tan t}+C.$$

21. $\int \frac{(\ln x)^2\,dx}{x}, \quad u=\ln x$

SOLUTION Let $u=\ln x$. Then $du=\frac{1}{x}\,dx$, and

$$\int \frac{(\ln x)^2}{x}\,dx=\int u^2\,du=\frac{1}{3}u^3+C=\frac{1}{3}(\ln x)^3+C.$$

22. $\int \frac{(\tan^{-1}x)^2\,dx}{x^2+1}, \quad u=\tan^{-1}x$

SOLUTION Let $u=\tan^{-1}x$. Then $du=\frac{1}{1+x^2}\,dx$, and

$$\int \frac{(\tan^{-1}x)^2}{x^2+1}\,dx=\int u^2\,du=\frac{1}{3}u^3+C=\frac{1}{3}(\tan^{-1}x)^3+C.$$

In Exercises 23–26, evaluate the integral in the form $a\sin(u(x))+C$ for an appropriate choice of $u(x)$ and constant a.

23. $\int x^3\cos(x^4)\,dx$

SOLUTION Let $u=x^4$. Then $du=4x^3\,dx$ or $\frac{1}{4}\,du=x^3dx$. Hence

$$\int x^3\cos(x^4)\,dx=\frac{1}{4}\int \cos u\,du=\frac{1}{4}\sin u+C=\frac{1}{4}\sin(x^4)+C.$$

24. $\int x^2\cos(x^3+1)\,dx$

SOLUTION Let $u=x^3+1$. Then $du=3x^2\,dx$ or $\frac{1}{3}\,du=x^2\,dx$. Hence

$$\int x^2\cos(x^3+1)\,dx=\frac{1}{3}\int \cos u\,du=\frac{1}{3}\sin u+C.$$

25. $\int x^{1/2}\cos(x^{3/2})\,dx$

SOLUTION Let $u=x^{3/2}$. Then $du=\frac{3}{2}x^{1/2}\,dx$ or $\frac{2}{3}\,du=x^{1/2}\,dx$. Hence

$$\int x^{1/2}\cos(x^{3/2})\,dx=\frac{2}{3}\int \cos u\,du=\frac{2}{3}\sin u+C=\frac{2}{3}\sin(x^{3/2})+C.$$

26. $\int \cos x \cos(\sin x)\, dx$

SOLUTION Let $u = \sin x$. Then $du = \cos x\, dx$. Hence

$$\int \cos x \cos(\sin x)\, dx = \int \cos u\, du = \sin u + C.$$

In Exercises 27–72, evaluate the indefinite integral.

27. $\int (4x + 5)^9\, dx$

SOLUTION Let $u = 4x + 5$. Then $du = 4\, dx$ and

$$\int (4x + 5)^9\, dx = \frac{1}{4}\int u^9\, du = \frac{1}{40}u^{10} + C = \frac{1}{40}(4x + 5)^{10} + C.$$

28. $\int \frac{dx}{(x - 9)^5}$

SOLUTION Let $u = x - 9$. Then $du = dx$ and

$$\int \frac{dx}{(x - 9)^5} = \int u^{-5}\, du = -\frac{1}{4}u^{-4} + C = -\frac{1}{4(x - 9)^4} + C.$$

29. $\int \frac{dt}{\sqrt{t + 12}}$

SOLUTION Let $u = t + 12$. Then $du = dt$ and

$$\int \frac{dt}{\sqrt{t + 12}} = \int u^{-1/2}\, du = 2u^{1/2} + C = 2\sqrt{t + 12} + C.$$

30. $\int (9t + 2)^{2/3}\, dt$

SOLUTION Let $u = 9t + 2$. Then $du = 9\, dt$ and

$$\int (9t + 2)^{2/3}\, dt = \frac{1}{9}\int u^{2/3}\, du = \frac{1}{9}\cdot\frac{3}{5}u^{5/3} + C = \frac{1}{15}(9t + 2)^{5/3} + C.$$

31. $\int \frac{x + 1}{(x^2 + 2x)^3}\, dx$

SOLUTION Let $u = x^2 + 2x$. Then $du = (2x + 2)\, dx$ or $\frac{1}{2}du = (x + 1)\, dx$. Hence

$$\int \frac{x + 1}{(x^2 + 2x)^3}\, dx = \frac{1}{2}\int \frac{1}{u^3}\, du = \frac{1}{2}\left(-\frac{1}{2}u^{-2}\right) + C = -\frac{1}{4}(x^2 + 2x)^{-2} + C = \frac{-1}{4(x^2 + 2x)^2} + C.$$

32. $\int (x + 1)(x^2 + 2x)^{3/4}\, dx$

SOLUTION Let $u = x^2 + 2x$. Then $du = (2x + 2)\, dx = 2(x + 1)\, dx$ and

$$\int (x + 1)(x^2 + 2x)^{3/4}\, dx = \frac{1}{2}\int u^{3/4}\, du = \frac{1}{2}\cdot\frac{4}{7}u^{7/4} + C$$
$$= \frac{2}{7}(x^2 + 2x)^{7/4} + C.$$

33. $\int \frac{x}{\sqrt{x^2 + 9}}\, dx$

SOLUTION Let $u = x^2 + 9$. Then $du = 2x\, dx$ or $\frac{1}{2}du = x\, dx$. Hence

$$\int \frac{x}{\sqrt{x^2 + 9}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du = \frac{1}{2}\frac{\sqrt{u}}{\frac{1}{2}} + C = \sqrt{x^2 + 9} + C.$$

34. $\int \frac{2x^2+x}{(4x^3+3x^2)^2}\,dx$

SOLUTION Let $u = 4x^3+3x^2$. Then $du = (12x^2+6x)\,dx$ or $\frac{1}{6}du = (2x^2+x)\,dx$. Hence

$$\int (4x^3+3x^2)^{-2}(2x^2+x)\,dx = \frac{1}{6}\int u^{-2}\,du = -\frac{1}{6}u^{-1}+C = -\frac{1}{6}(4x^3+3x^2)^{-1}+C.$$

35. $\int (3x^2+1)(x^3+x)^2\,dx$

SOLUTION Let $u = x^3+x$. Then $du = (3x^2+1)\,dx$. Hence

$$\int (3x^2+1)(x^3+x)^2\,dx = \int u^2\,du = \frac{1}{3}u^3+C = \frac{1}{3}(x^3+x)^3+C.$$

36. $\int \frac{5x^4+2x}{(x^5+x^2)^3}\,dx$

SOLUTION Let $u = x^5+x^2$. Then $du = (5x^4+2x)\,dx$. Hence

$$\int \frac{5x^4+2x}{(x^5+x^2)^3}\,dx = \int \frac{1}{u^3}\,du = -\frac{1}{2}\frac{1}{u^2}+C = -\frac{1}{2}\frac{1}{(x^5+x^2)^2}+C.$$

37. $\int (3x+8)^{11}\,dx$

SOLUTION Let $u = 3x+8$. Then $du = 3\,dx$ and

$$\int (3x+8)^{11}\,dx = \frac{1}{3}\int u^{11}\,du = \frac{1}{36}u^{12}+C = \frac{1}{36}(3x+8)^{12}+C.$$

38. $\int x(3x+8)^{11}\,dx$

SOLUTION Let $u = 3x+8$. Then $du = 3\,dx$, $x = \frac{u-8}{3}$, and

$$\begin{aligned}\int x(3x+8)^{11}\,dx &= \frac{1}{9}\int (u-8)u^{11}\,du = \frac{1}{9}\int (u^{12}-8u^{11})\,du\\ &= \frac{1}{9}\left(\frac{1}{13}u^{13}-\frac{2}{3}u^{12}\right)+C\\ &= \frac{1}{117}(3x+8)^{13}-\frac{2}{27}(3x+8)^{12}+C.\end{aligned}$$

39. $\int x^2\sqrt{x^3+1}\,dx$

SOLUTION Let $u = x^3+1$. Then $du = 3x^2\,dx$ and

$$\int x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int u^{1/2}\,du = \frac{2}{9}u^{3/2}+C = \frac{2}{9}(x^3+1)^{3/2}+C.$$

40. $\int x^5\sqrt{x^3+1}\,dx$

SOLUTION Let $u = x^3+1$. Then $du = 3x^2\,dx$, $x^3 = u-1$ and

$$\begin{aligned}\int x^5\sqrt{x^3+1}\,dx &= \frac{1}{3}\int (u-1)\sqrt{u}\,du = \frac{1}{3}\int (u^{3/2}-u^{1/2})\,du\\ &= \frac{1}{3}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)+C\\ &= \frac{2}{15}(x^3+1)^{5/2}-\frac{2}{9}(x^3+1)^{3/2}+C.\end{aligned}$$

41. $\int \frac{dx}{(x+5)^3}$

SOLUTION Let $u = x+5$. Then $du = dx$ and

$$\int \frac{dx}{(x+5)^3} = \int u^{-3}\,du = -\frac{1}{2}u^{-2}+C = -\frac{1}{2}(x+5)^{-2}+C.$$

42. $\displaystyle\int \frac{x^2\,dx}{(x+5)^3}$

SOLUTION Let $u = x + 5$. Then $du = dx$, $x = u - 5$ and

$$\int \frac{x^2\,dx}{(x+5)^3} = \int \frac{(u-5)^2}{u^3}\,du = \int (u^{-1} - 10u^{-2} + 25u^{-3})\,du$$
$$= \ln|u| + 10u^{-1} - \frac{25}{2}u^{-2} + C$$
$$= \ln|x+5| + \frac{10}{x+5} - \frac{25}{2(x+5)^2} + C.$$

43. $\displaystyle\int z^2(z^3+1)^{12}\,dz$

SOLUTION Let $u = z^3 + 1$. Then $du = 3z^2\,dz$ and

$$\int z^2(z^3+1)^{12}\,dz = \frac{1}{3}\int u^{12}\,du = \frac{1}{39}u^{13} + C = \frac{1}{39}(z^3+1)^{13} + C.$$

44. $\displaystyle\int (z^5 + 4z^2)(z^3+1)^{12}\,dz$

SOLUTION Let $u = z^3 + 1$. Then $du = 3z^2\,dz$, $z^3 = u - 1$ and

$$\int (z^5 + 4z^2)(z^3+1)^{12}\,dz = \frac{1}{3}\int (u+3)u^{12}\,du = \frac{1}{3}\int (u^{13} + 3u^{12})\,du$$
$$= \frac{1}{3}\left(\frac{1}{14}u^{14} + \frac{3}{13}u^{13}\right) + C$$
$$= \frac{1}{42}(z^3+1)^{14} + \frac{1}{13}(z^3+1)^{13} + C.$$

45. $\displaystyle\int (x+2)(x+1)^{1/4}\,dx$

SOLUTION Let $u = x + 1$. Then $x = u - 1$, $du = dx$ and

$$\int (x+2)(x+1)^{1/4}\,dx = \int (u+1)u^{1/4}\,du = \int (u^{5/4} + u^{1/4})\,du$$
$$= \frac{4}{9}u^{9/4} + \frac{4}{5}u^{5/4} + C$$
$$= \frac{4}{9}(x+1)^{9/4} + \frac{4}{5}(x+1)^{5/4} + C.$$

46. $\displaystyle\int x^3(x^2-1)^{3/2}\,dx$

SOLUTION Let $u = x^2 - 1$. Then $u + 1 = x^2$ and $du = 2x\,dx$ or $\frac{1}{2}\,du = x\,dx$. Hence

$$\int x^3(x^2-1)^{3/2}\,dx = \int x^2 \cdot x(x^2-1)^{3/2}\,dx$$
$$= \frac{1}{2}\int (u+1)u^{3/2}\,du = \frac{1}{2}\int (u^{5/2} + u^{3/2})\,du$$
$$= \frac{1}{2}\left(\frac{2}{7}u^{7/2}\right) + \frac{1}{2}\left(\frac{2}{5}u^{5/2}\right) + C = \frac{1}{7}(x^2-1)^{7/2} + \frac{1}{5}(x^2-1)^{5/2} + C.$$

47. $\displaystyle\int \sin(8 - 3\theta)\,d\theta$

SOLUTION Let $u = 8 - 3\theta$. Then $du = -3\,d\theta$ and

$$\int \sin(8-3\theta)\,d\theta = -\frac{1}{3}\int \sin u\,du = \frac{1}{3}\cos u + C = \frac{1}{3}\cos(8-3\theta) + C.$$

48. $\int \theta \sin(\theta^2)\, d\theta$

SOLUTION Let $u = \theta^2$. Then $du = 2\theta\, d\theta$ and

$$\int \theta \sin(\theta^2)\, d\theta = \frac{1}{2}\int \sin u\, du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(\theta^2) + C.$$

49. $\int \frac{\cos\sqrt{t}}{\sqrt{t}}\, dt$

SOLUTION Let $u = \sqrt{t} = t^{1/2}$. Then $du = \frac{1}{2}t^{-1/2}\, dt$ and

$$\int \frac{\cos\sqrt{t}}{\sqrt{t}}\, dt = 2\int \cos u\, du = 2\sin u + C = 2\sin\sqrt{t} + C.$$

50. $\int x^2 \sin(x^3+1)\, dx$

SOLUTION Let $u = x^3 + 1$. Then $du = 3x^2\, dx$ or $\frac{1}{3}du = x^2\, dx$. Hence

$$\int x^2 \sin(x^3+1)\, dx = \frac{1}{3}\int \sin u\, du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos(x^3+1) + C.$$

51. $\int \tan(4\theta + 9)\, d\theta$

SOLUTION Let $u = 4\theta + 9$. Then $du = 4\, d\theta$ and

$$\int \tan(4\theta+9)\, d\theta = \frac{1}{4}\int \tan u\, du = \frac{1}{4}\ln|\sec u| + C = \frac{1}{4}\ln|\sec(4\theta+9)| + C.$$

52. $\int \sin^8\theta \cos\theta\, d\theta$

SOLUTION Let $u = \sin\theta$. Then $du = \cos\theta\, d\theta$ and

$$\int \sin^8\theta\cos\theta\, d\theta = \int u^8\, du = \frac{1}{9}u^9 + C = \frac{1}{9}\sin^9\theta + C.$$

53. $\int \cot x\, dx$

SOLUTION Let $u = \sin x$. Then $du = \cos x\, dx$, and

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C.$$

54. $\int x^{-1/5}\tan x^{4/5}\, dx$

SOLUTION Let $u = x^{4/5}$. Then $du = \frac{4}{5}x^{-1/5}\, dx$ and

$$\int x^{-1/5}\tan x^{4/5}\, dx = \frac{5}{4}\int \tan u\, du = \frac{5}{4}\ln|\sec u| + C = \frac{5}{4}\ln|\sec x^{4/5}| + C.$$

55. $\int \sec^2(4x+9)\, dx$

SOLUTION Let $u = 4x + 9$. Then $du = 4\, dx$ or $\frac{1}{4}\, du = dx$. Hence

$$\int \sec^2(4x+9)\, dx = \frac{1}{4}\int \sec^2 u\, du = \frac{1}{4}\tan u + C = \frac{1}{4}\tan(4x+9) + C.$$

56. $\int \sec^2 x \tan^4 x\, dx$

SOLUTION Let $u = \tan x$. Then $du = \sec^2 x\, dx$. Hence

$$\int \sec^2 x\tan^4 x\, dx = \int u^4\, du = \frac{1}{5}u^5 + C = \frac{1}{5}\tan^5 x + C.$$

57. $\displaystyle\int \frac{\sec^2(\sqrt{x})\,dx}{\sqrt{x}}$

SOLUTION Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$ or $2\,du = \frac{1}{\sqrt{x}}\,dx$. Hence,

$$\int \frac{\sec^2(\sqrt{x})\,dx}{\sqrt{x}} = 2\int \sec^2 u\,dx = 2\tan u + C = 2\tan(\sqrt{x}) + C.$$

58. $\displaystyle\int \frac{\cos 2x}{(1+\sin 2x)^2}\,dx$

SOLUTION Let $u = 1 + \sin 2x$. Then $du = 2\cos 2x$ or $\frac{1}{2}du = \cos 2x\,dx$. Hence

$$\int (1+\sin 2x)^{-2}\cos 2x\,dx = \frac{1}{2}\int u^{-2}\,du = -\frac{1}{2}u^{-1} + C = -\frac{1}{2}(1+\sin 2x)^{-1} + C.$$

59. $\displaystyle\int \sin 4x\sqrt{\cos 4x + 1}\,dx$

SOLUTION Let $u = \cos 4x + 1$. Then $du = -4\sin 4x$ or $-\frac{1}{4}du = \sin 4x$. Hence

$$\int \sin 4x\sqrt{\cos 4x+1}\,dx = -\frac{1}{4}\int u^{1/2}\,du = -\frac{1}{4}\left(\frac{2}{3}u^{3/2}\right) + C = -\frac{1}{6}(\cos 4x + 1)^{3/2} + C.$$

60. $\displaystyle\int \cos x(3\sin x - 1)\,dx$

SOLUTION Let $u = 3\sin x - 1$. Then $du = 3\cos x\,dx$ or $\frac{1}{3}du = \cos x\,dx$. Hence

$$\int (3\sin x - 1)\cos x\,dx = \frac{1}{3}\int u\,du = \frac{1}{3}\left(\frac{1}{2}u^2\right) + C = \frac{1}{6}(3\sin x - 1)^2 + C.$$

61. $\displaystyle\int \sec\theta\tan\theta(\sec\theta - 1)\,d\theta$

SOLUTION Let $u = \sec\theta - 1$. Then $du = \sec\theta\tan\theta\,d\theta$ and

$$\int \sec\theta\tan\theta(\sec\theta - 1)\,d\theta = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\sec\theta - 1)^2 + C.$$

62. $\displaystyle\int \cos t\cos(\sin t)\,dt$

SOLUTION Let $u = \sin t$. Then $du = \cos t\,dt$ and

$$\int \cos t\cos(\sin t)\,dt = \int \cos u\,du = \sin u + C = \sin(\sin t) + C.$$

63. $\displaystyle\int e^{14x-7}\,dx$

SOLUTION Let $u = 14x - 7$. Then $du = 14\,dx$ or $\frac{1}{14}\,du = dx$. Hence,

$$\int e^{14x-7}\,dx = \frac{1}{14}\int e^u\,du = \frac{1}{14}e^u + C = \frac{1}{14}e^{14x-7} + C.$$

64. $\displaystyle\int (x+1)e^{x^2+2x}\,dx$

SOLUTION Let $u = x^2 + 2x$. Then $du = (2x+2)\,dx$ or $\frac{1}{2}\,du = (x+1)\,dx$. Hence,

$$\int (x+1)e^{x^2+2x}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2+2x} + C.$$

65. $\displaystyle\int \frac{e^x\,dx}{(e^x+1)^4}$

SOLUTION Let $u = e^x + 1$. Then $du = e^x\,dx$, and

$$\int \frac{e^x}{(e^x+1)^4}\,dx = \int u^{-4}\,du = -\frac{1}{3u^3} + C = -\frac{1}{3(e^x+1)^3} + C.$$

66. $\int (\sec^2\theta)\, e^{\tan\theta}\, d\theta$

SOLUTION Let $u = \tan\theta$. Then $du = \sec^2\theta\, d\theta$, and

$$\int (\sec^2\theta)\, e^{\tan\theta}\, d\theta = \int e^u\, du = e^u + C = e^{\tan\theta} + C.$$

67. $\int \frac{e^t\, dt}{e^{2t} + 2e^t + 1}$

SOLUTION Let $u = e^t$. Then $du = e^t\, dt$, and

$$\int \frac{e^t\, dt}{e^{2t} + 2e^t + 1} = \int \frac{du}{u^2 + 2u + 1} = \int \frac{du}{(u+1)^2} = -\frac{1}{u+1} + C = -\frac{1}{e^t + 1} + C.$$

68. $\int \frac{dx}{x(\ln x)^2}$

SOLUTION Let $u = \ln x$. Then $du = \frac{1}{x}\, dx$, and

$$\int \frac{dx}{x(\ln x)^2} = \int u^{-2}\, du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C.$$

69. $\int \frac{(\ln x)^4\, dx}{x}$

SOLUTION Let $u = \ln x$. Then $du = \frac{1}{x}\, dx$, and

$$\int \frac{(\ln x)^4}{x}\, dx = \int u^4\, du = \frac{1}{5}u^5 + C = \frac{1}{5}(\ln x)^5 + C.$$

70. $\int \frac{dx}{x \ln x}$

SOLUTION Let $u = \ln x$. Then $du = \frac{1}{x}\, dx$, and

$$\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C.$$

71. $\int \frac{\tan(\ln x)}{x}\, dx$

SOLUTION Let $u = \cos(\ln x)$. Then $du = -\frac{1}{x}\sin(\ln x)\, dx$ or $-du = \frac{1}{x}\sin(\ln x)\, dx$. Hence,

$$\int \frac{\tan(\ln x)}{x}\, dx = \int \frac{\sin(\ln x)}{x\cos(\ln x)}\, dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos(\ln x)| + C.$$

72. $\int (\cot x)\ln(\sin x)\, dx$

SOLUTION Let $u = \ln(\sin x)$. Then

$$du = \frac{1}{\sin x}\cos x = \cot x,$$

and

$$\int (\cot x)\ln(\sin x)\, dx = \int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln(\sin x))^2 + C.$$

73. Evaluate $\int \frac{dx}{(1+\sqrt{x})^3}$ using $u = 1 + \sqrt{x}$. *Hint:* Show that $dx = 2(u-1)du$.

SOLUTION Let $u = 1 + \sqrt{x}$. Then

$$du = \frac{1}{2\sqrt{x}}\, dx \quad \text{or} \quad dx = 2\sqrt{x}\, du = 2(u-1)\, du.$$

Hence,

$$\int \frac{dx}{(1+\sqrt{x})^3} = 2\int \frac{u-1}{u^3}\,du = 2\int (u^{-2}-u^{-3})\,du$$
$$= -2u^{-1} + u^{-2} + C = -\frac{2}{1+\sqrt{x}} + \frac{1}{(1+\sqrt{x})^2} + C.$$

74. Can They Both Be Right? Hannah uses the substitution $u = \tan x$ and Akiva uses $u = \sec x$ to evaluate $\int \tan x \sec^2 x\,dx$. Show that they obtain different answers, and explain the apparent contradiction.

SOLUTION With the substitution $u = \tan x$, Hannah finds $du = \sec^2 x\,dx$ and

$$\int \tan x \sec^2 x\,dx = \int u\,du = \frac{1}{2}u^2 + C_1 = \frac{1}{2}\tan^2 x + C_1.$$

On the other hand, with the substitution $u = \sec x$, Akiva finds $du = \sec x \tan x\,dx$ and

$$\int \tan x \sec^2 x\,dx = \int \sec x(\tan x \sec x)\,dx = \frac{1}{2}\sec^2 x + C_2$$

Hannah and Akiva have each found a correct antiderivative. To resolve what appears to be a contradiction, recall that any two antiderivatives of a specified function differ by a constant. To show that this is true in their case, note that

$$\left(\frac{1}{2}\sec^2 x + C_2\right) - \left(\frac{1}{2}\tan^2 x + C_1\right) = \frac{1}{2}(\sec^2 x - \tan^2 x) + C_2 - C_1$$
$$= \frac{1}{2}(1) + C_2 - C_1 = \frac{1}{2} + C_2 - C_1, \text{ a constant}$$

Here we used the trigonometric identity $\tan^2 x + 1 = \sec^2 x$.

75. Evaluate $\int \sin x \cos x\,dx$ using substitution in two different ways: first using $u = \sin x$ and then using $u = \cos x$. Reconcile the two different answers.

SOLUTION First, let $u = \sin x$. Then $du = \cos x\,dx$ and

$$\int \sin x \cos x\,dx = \int u\,du = \frac{1}{2}u^2 + C_1 = \frac{1}{2}\sin^2 x + C_1.$$

Next, let $u = \cos x$. Then $du = -\sin x\,dx$ or $-du = \sin x\,dx$. Hence,

$$\int \sin x \cos x\,dx = -\int u\,du = -\frac{1}{2}u^2 + C_2 = -\frac{1}{2}\cos^2 x + C_2.$$

To reconcile these two seemingly different answers, recall that any two antiderivatives of a specified function differ by a constant. To show that this is true here, note that $(\frac{1}{2}\sin^2 x + C_1) - (-\frac{1}{2}\cos^2 x + C_2) = \frac{1}{2} + C_1 - C_2$, a constant. Here we used the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

76. Some Choices Are Better Than Others Evaluate

$$\int \sin x\ \cos^2 x\,dx$$

twice. First use $u = \sin x$ to show that

$$\int \sin x\ \cos^2 x\,dx = \int u\sqrt{1-u^2}\,du$$

and evaluate the integral on the right by a further substitution. Then show that $u = \cos x$ is a better choice.

SOLUTION Consider the integral $\int \sin x \cos^2 x\,dx$. If we let $u = \sin x$, then $\cos x = \sqrt{1-u^2}$ and $du = \cos x\,dx$. Hence

$$\int \sin x \cos^2 x\,dx = \int u\sqrt{1-u^2}\,du.$$

Now let $w = 1 - u^2$. Then $dw = -2u\,du$ or $-\frac{1}{2}dw = u\,du$. Therefore,

$$\int u\sqrt{1-u^2}\,du = -\frac{1}{2}\int w^{1/2}\,dw = -\frac{1}{2}\left(\frac{2}{3}w^{3/2}\right) + C$$
$$= -\frac{1}{3}w^{3/2} + C = -\frac{1}{3}(1-u^2)^{3/2} + C$$
$$= -\frac{1}{3}(1-\sin^2 x)^{3/2} + C = -\frac{1}{3}\cos^3 x + C.$$

A better substitution choice is $u = \cos x$. Then $du = -\sin x\,dx$ or $-du = \sin x\,dx$. Hence

$$\int \sin x \cos^2 x\,dx = -\int u^2\,du = -\frac{1}{3}u^3 + C = -\frac{1}{3}\cos^3 x + C.$$

77. What are the new limits of integration if we apply the substitution $u = 3x + \pi$ to the integral $\int_0^\pi \sin(3x + \pi)\,dx$?

SOLUTION The new limits of integration are $u(0) = 3 \cdot 0 + \pi = \pi$ and $u(\pi) = 3\pi + \pi = 4\pi$.

78. Which of the following is the result of applying the substitution $u = 4x - 9$ to the integral $\int_2^8 (4x-9)^{20}\,dx$?

(a) $\displaystyle\int_2^8 u^{20}\,du$

(b) $\displaystyle\frac{1}{4}\int_2^8 u^{20}\,du$

(c) $\displaystyle 4\int_{-1}^{23} u^{20}\,du$

(d) $\displaystyle\frac{1}{4}\int_{-1}^{23} u^{20}\,du$

SOLUTION Let $u = 4x - 9$. Then $du = 4\,dx$ or $\frac{1}{4}\,du = dx$. Furthermore, when $x = 2$, $u = -1$, and when $x = 8$, $u = 23$. Hence

$$\int_2^8 (4x-9)^{20}\,dx = \frac{1}{4}\int_{-1}^{23} u^{20}\,du.$$

The answer is therefore **(d)**.

In Exercises 79–90, use the Change-of-Variables Formula to evaluate the definite integral.

79. $\displaystyle\int_1^3 (x+2)^3\,dx$

SOLUTION Let $u = x + 2$. Then $du = dx$. Hence

$$\int_1^3 (x+2)^3\,dx = \int_3^5 u^3\,du = \left.\frac{1}{4}u^4\right|_3^5 = \frac{5^4}{4} - \frac{3^4}{4} = 136.$$

80. $\displaystyle\int_1^6 \sqrt{x+3}\,dx$

SOLUTION Let $u = x + 3$. Then $du = dx$. Hence

$$\int_1^6 \sqrt{x+3}\,dx = \int_4^9 \sqrt{u}\,du = \left.\frac{2}{3}u^{3/2}\right|_4^9 = \frac{2}{3}(27-8) = \frac{38}{3}.$$

81. $\displaystyle\int_0^1 \frac{x}{(x^2+1)^3}\,dx$

SOLUTION Let $u = x^2 + 1$. Then $du = 2x\,dx$ or $\frac{1}{2}\,du = x\,dx$. Hence

$$\int_0^1 \frac{x}{(x^2+1)^3}\,dx = \frac{1}{2}\int_1^2 \frac{1}{u^3}\,du = \left.\frac{1}{2}\left(-\frac{1}{2}u^{-2}\right)\right|_1^2 = -\frac{1}{16} + \frac{1}{4} = \frac{3}{16} = 0.1875.$$

82. $\displaystyle\int_{-1}^2 \sqrt{5x+6}\,dx$

SOLUTION Let $u = 5x + 6$. Then $du = 5\,dx$ or $\frac{1}{5}\,du = dx$. Hence

$$\int_{-1}^2 \sqrt{5x+6}\,dx = \frac{1}{5}\int_1^{16} \sqrt{u}\,du = \left.\frac{1}{5}\left(\frac{2}{3}u^{3/2}\right)\right|_1^{16} = \frac{2}{15}(64-1) = \frac{42}{5}.$$

83. $\displaystyle\int_0^4 x\sqrt{x^2+9}\,dx$

SOLUTION Let $u = x^2 + 9$. Then $du = 2x\,dx$ or $\frac{1}{2}\,du = x\,dx$. Hence

$$\int_0^4 \sqrt{x^2+9}\,dx = \frac{1}{2}\int_9^{25} \sqrt{u}\,du = \left.\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\right|_9^{25} = \frac{1}{3}(125-27) = \frac{98}{3}.$$

84. $\int_1^2 \frac{4x+12}{(x^2+6x+1)^2}\,dx$

SOLUTION Let $u = x^2 + 6x + 1$. Then $du = (2x+6)\,dx$ and

$$\int_1^2 \frac{4x+12}{(x^2+6x+1)^2}\,dx = 2\int_8^{17} u^{-2}\,du = -\frac{2}{u}\Big|_8^{17}$$
$$= -\frac{2}{17} + \frac{1}{4} = \frac{9}{68}.$$

85. $\int_0^1 (x+1)(x^2+2x)^5\,dx$

SOLUTION Let $u = x^2 + 2x$. Then $du = (2x+2)\,dx = 2(x+1)\,dx$, and

$$\int_0^1 (x+1)(x^2+2x)^5\,dx = \frac{1}{2}\int_0^3 u^5\,du = \frac{1}{12}u^6\Big|_0^3 = \frac{729}{12} = \frac{243}{4}.$$

86. $\int_{10}^{17} (x-9)^{-2/3}\,dx$

SOLUTION Let $u = x - 9$. Then $du = dx$. Hence

$$\int_{10}^{17} (x-9)^{-2/3}\,dx = \int_1^8 u^{-2/3}\,dx = 3u^{1/3}\Big|_1^8 = 3\,(2-1) = 3.$$

87. $\int_0^1 \theta\tan(\theta^2)\,d\theta$

SOLUTION Let $u = \cos\theta^2$. Then $du = -2\theta\sin\theta^2\,d\theta$ or $-\frac{1}{2}du = \theta\sin\theta^2\,d\theta$. Hence,

$$\int_0^1 \theta\tan(\theta^2)\,d\theta = \int_0^1 \frac{\theta\sin(\theta^2)}{\cos(\theta^2)}\,d\theta = -\frac{1}{2}\int_1^{\cos 1}\frac{du}{u} = -\frac{1}{2}\ln|u|\Big|_1^{\cos 1} = -\frac{1}{2}\left[\ln(\cos 1) + \ln 1\right] = \frac{1}{2}\ln(\sec 1).$$

88. $\int_0^{\pi/6} \sec^2\left(2x - \frac{\pi}{6}\right)\,dx$

SOLUTION Let $u = 2x - \frac{\pi}{6}$. Then $du = 2\,dx$ and

$$\int_0^{\pi/6} \sec^2\left(2x - \frac{\pi}{6}\right)\,dx = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \sec^2 u\,du = \frac{1}{2}\tan u\Big|_{-\pi/6}^{\pi/6}$$
$$= \frac{1}{2}\left(\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3}.$$

89. $\int_0^{\pi/2} \cos^3 x\sin x\,dx$

SOLUTION Let $u = \cos x$. Then $du = -\sin x\,dx$. Hence

$$\int_0^{\pi/2} \cos^3 x\sin x\,dx = -\int_1^0 u^3\,du = \int_0^1 u^3\,du = \frac{1}{4}u^4\Big|_0^1 = \frac{1}{4} - 0 = \frac{1}{4}.$$

90. $\int_{\pi/3}^{\pi/2} \cot^2\frac{x}{2}\csc^2\frac{x}{2}\,dx$

SOLUTION Let $u = \cot\frac{x}{2}$. Then $du = -\frac{1}{2}\csc^2\frac{x}{2}$ and

$$\int_{\pi/3}^{\pi/2} \cot^2\frac{x}{2}\csc^2\frac{x}{2}\,dx = -2\int_{\sqrt{3}}^1 u^2\,du$$
$$= -\frac{2}{3}u^3\Big|_{\sqrt{3}}^1 = \frac{2}{3}(3\sqrt{3} - 1).$$

91. Evaluate $\int_0^2 r\sqrt{5-\sqrt{4-r^2}}\,dr$.

SOLUTION Let $u = 5 - \sqrt{4-r^2}$. Then

$$du = \frac{r\,dr}{\sqrt{4-r^2}} = \frac{r\,dr}{5-u}$$

so that

$$r\,dr = (5-u)\,du.$$

Hence, the integral becomes:

$$\int_0^2 r\sqrt{5-\sqrt{4-r^2}}\,dr = \int_3^5 \sqrt{u}(5-u)\,du = \int_3^5 \left(5u^{1/2} - u^{3/2}\right)\,du = \left(\frac{10}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_3^5$$

$$= \left(\frac{50}{3}\sqrt{5} - 10\sqrt{5}\right) - \left(10\sqrt{3} - \frac{18}{5}\sqrt{3}\right) = \frac{20}{3}\sqrt{5} - \frac{32}{5}\sqrt{3}.$$

92. Find numbers a and b such that

$$\int_a^b (u^2+1)\,du = \int_{-\pi/4}^{\pi/4} \sec^4\theta\,d\theta$$

and evaluate. *Hint:* Use the identity $\sec^2\theta = \tan^2\theta + 1$.

SOLUTION Let $u = \tan\theta$. Then $u^2+1 = \tan^2\theta + 1 = \sec^2\theta$ and $du = \sec^2\theta\,d\theta$. Moreover, because

$$\tan\left(-\frac{\pi}{4}\right) = -1 \quad \text{and} \quad \tan\frac{\pi}{4} = 1,$$

it follows that $a = -1$ and $b = 1$. Thus,

$$\int_{-\pi/4}^{\pi/4} \sec^4\theta\,d\theta = \int_{-1}^{1} (u^2+1)\,du = \left(\frac{1}{3}u^3 + u\right)\Big|_{-1}^{1} = \frac{8}{3}.$$

93. Wind engineers have found that wind speed v (in meters/second) at a given location follows a **Rayleigh distribution** of the type

$$W(v) = \frac{1}{32}ve^{-v^2/64}$$

This means that at a given moment in time, the probability that v lies between a and b is equal to the shaded area in Figure 4.

(a) Show that the probability that $v \in [0, b]$ is $1 - e^{-b^2/64}$.

(b) Calculate the probability that $v \in [2, 5]$.

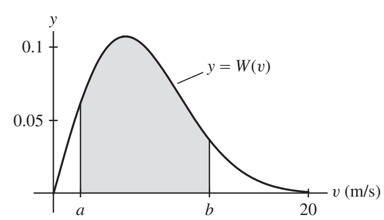


FIGURE 4 The shaded area is the probability that v lies between$a$ and b.

SOLUTION

(a) The probability that $v \in [0, b]$ is

$$\int_0^b \frac{1}{32}ve^{-v^2/64}\,dv.$$

Let $u = -v^2/64$. Then $du = -v/32\,dv$ and

$$\int_0^b \frac{1}{32}ve^{-v^2/64}\,dv = -\int_0^{-b^2/64} e^u\,du = -e^u\Big|_0^{-b^2/64} = -e^{-b^2/64} + 1.$$

(b) The probability that $v \in [2, 5]$ is the probability that $v \in [0, 5]$ minus the probability that $v \in [0, 2]$. By part (a), the probability that $v \in [2, 5]$ is

$$\left(1 - e^{-25/64}\right) - \left(1 - e^{-1/16}\right) = e^{-1/16} - e^{-25/64}.$$

94. Evaluate $\int_0^{\pi/2} \sin^n x \cos x\, dx$ for $n \geq 0$.

SOLUTION Let $u = \sin x$. Then $du = \cos x\, dx$. Hence

$$\int_0^{\pi/2} \sin^n x\ \cos x\, dx = \int_0^1 u^n\, du = \left.\frac{u^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}.$$

In Exercises 95–96, use substitution to evaluate the integral in terms of $f(x)$.

95. $\int f(x)^3 f'(x)\, dx$

SOLUTION Let $u = f(x)$. Then $du = f'(x)\, dx$. Hence

$$\int f(x)^3\ f'(x)\, dx = \int u^3\, du = \frac{1}{4}u^4 + C = \frac{1}{4}f(x)^4 + C.$$

96. $\int \frac{f'(x)}{f(x)^2}\, dx$

SOLUTION Let $u = f(x)$. Then $du = f'(x)\, dx$. Hence

$$\int \frac{f'(x)}{f(x)^2}\, dx = \int u^{-2}\, du = -u^{-1} + C = \frac{-1}{f(x)} + C.$$

97. Show that $\int_0^{\pi/6} f(\sin\theta)\, d\theta = \int_0^{1/2} f(u)\frac{1}{\sqrt{1-u^2}}\, du$.

SOLUTION Let $u = \sin\theta$. Then $u(\pi/6) = 1/2$ and $u(0) = 0$, as required. Furthermore, $du = \cos\theta\, d\theta$, so that

$$d\theta = \frac{du}{\cos\theta}.$$

If $\sin\theta = u$, then $u^2 + \cos^2\theta = 1$, so that $\cos\theta = \sqrt{1-u^2}$. Therefore $d\theta = du/\sqrt{1-u^2}$. This gives

$$\int_0^{\pi/6} f(\sin\theta)\, d\theta = \int_0^{1/2} f(u)\frac{1}{\sqrt{1-u^2}}\, du.$$

Further Insights and Challenges

98. Use the substitution $u = 1 + x^{1/n}$ to show that

$$\int \sqrt{1+x^{1/n}}\, dx = n\int u^{1/2}(u-1)^{n-1}\, du$$

Evaluate for $n = 2, 3$.

SOLUTION Let $u = 1 + x^{1/n}$. Then $x = (u-1)^n$ and $dx = n(u-1)^{n-1}\, du$. Accordingly, $\int \sqrt{1+x^{1/n}}\, dx = n\int u^{1/2}(u-1)^{n-1}\, du$.

For $n = 2$, we have

$$\int \sqrt{1+x^{1/2}}\, dx = 2\int u^{1/2}(u-1)^1\, du = 2\int (u^{3/2} - u^{1/2})\, du$$
$$= 2\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C = \frac{4}{5}(1+x^{1/2})^{5/2} - \frac{4}{3}(1+x^{1/2})^{3/2} + C.$$

For $n = 3$, we have

$$\int \sqrt{1+x^{1/3}}\, dx = 3\int u^{1/2}(u-1)^2\, du = 3\int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du$$

$$= 3\left(\frac{2}{7}u^{7/2} - (2)\left(\frac{2}{5}\right)u^{5/2} + \frac{2}{3}u^{3/2}\right) + C$$
$$= \frac{6}{7}(1 + x^{1/3})^{7/2} - \frac{12}{5}(1 + x^{1/3})^{5/2} + 2(1 + x^{1/3})^{3/2} + C.$$

99. Evaluate $I = \int_0^{\pi/2} \frac{d\theta}{1 + \tan^{6,000}\theta}$. *Hint:* Use substitution to show that I is equal to $J = \int_0^{\pi/2} \frac{d\theta}{1 + \cot^{6,000}\theta}$ and then check that $I + J = \int_0^{\pi/2} d\theta$.

SOLUTION To evaluate

$$I = \int_0^{\pi/2} \frac{dx}{1 + \tan^{6000} x},$$

we substitute $t = \pi/2 - x$. Then $dt = -dx$, $x = \pi/2 - t$, $t(0) = \pi/2$, and $t(\pi/2) = 0$. Hence,

$$I = \int_0^{\pi/2} \frac{dx}{1 + \tan^{6000} x} = -\int_{\pi/2}^{0} \frac{dt}{1 + \tan^{6000}(\pi/2 - t)} = \int_0^{\pi/2} \frac{dt}{1 + \cot^{6000} t}.$$

Let $J = \int_0^{\pi/2} \frac{dt}{1 + \cot^{6000}(t)}$. We know $I = J$, so $I + J = 2I$. On the other hand, by the definition of I and J and the linearity of the integral,

$$I + J = \int_0^{\pi/2} \frac{dx}{1 + \tan^{6000} x} + \frac{dx}{1 + \cot^{6000} x} = \int_0^{\pi/2} \left(\frac{1}{1 + \tan^{6000} x} + \frac{1}{1 + \cot^{6000} x}\right) dx$$
$$= \int_0^{\pi/2} \left(\frac{1}{1 + \tan^{6000} x} + \frac{1}{1 + (1/\tan^{6000} x)}\right) dx$$
$$= \int_0^{\pi/2} \left(\frac{1}{1 + \tan^{6000} x} + \frac{1}{(\tan^{6000} x + 1)/\tan^{6000} x}\right) dx$$
$$= \int_0^{\pi/2} \left(\frac{1}{1 + \tan^{6000} x} + \frac{\tan^{6000} x}{1 + \tan^{6000} x}\right) dx$$
$$= \int_0^{\pi/2} \left(\frac{1 + \tan^{6000} x}{1 + \tan^{6000} x}\right) dx = \int_0^{\pi/2} 1\, dx = \pi/2.$$

Hence, $I + J = 2I = \pi/2$, so $I = \pi/4$.

100. Use substitution to prove that $\int_{-a}^{a} f(x)\, dx = 0$ if f is an odd function.

SOLUTION We assume that f is continuous. If $f(x)$ is an odd function, then $f(-x) = -f(x)$. Let $u = -x$. Then $x = -u$ and $du = -dx$ or $-du = dx$. Accordingly,

$$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx = -\int_a^0 f(-u)\, du + \int_0^a f(x)\, dx$$
$$= \int_0^a f(x)\, dx - \int_0^a f(u)\, du = 0.$$

101. Prove that $\int_a^b \frac{1}{x}\, dx = \int_1^{b/a} \frac{1}{x}\, dx$ for $a, b > 0$. Then show that the regions under the hyperbola over the intervals $[1, 2], [2, 4], [4, 8], \ldots$ all have the same area (Figure 5).

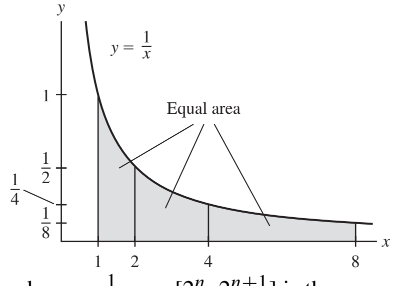


FIGURE 5 The area under $y = \frac{1}{x}$ over $[2^n, 2^{n+1}]$ is the same for all $n = 0, 1, 2, \ldots$.

SOLUTION

(a) Let $u = \frac{x}{a}$. Then $au = x$ and $du = \frac{1}{a}\,dx$ or $a\,du = dx$. Hence

$$\int_a^b \frac{1}{x}\,dx = \int_1^{b/a} \frac{a}{au}\,du = \int_1^{b/a} \frac{1}{u}\,du.$$

Note that $\int_1^{b/a} \frac{1}{u}\,du = \int_1^{b/a} \frac{1}{x}\,dx$ after the substitution $x = u$.

(b) The area under the hyperbola over the interval [1, 2] is given by the definite integral $\int_1^2 \frac{1}{x}\,dx$. Denote this definite integral by A. Using the result from part (a), we find the area under the hyperbola over the interval [2, 4] is

$$\int_2^4 \frac{1}{x}\,dx = \int_1^{4/2} \frac{1}{x}\,dx = \int_1^2 \frac{1}{x}\,dx = A.$$

Similarly, the area under the hyperbola over the interval [4, 8] is

$$\int_4^8 \frac{1}{x}\,dx = \int_1^{8/4} \frac{1}{x}\,dx = \int_1^2 \frac{1}{x}\,dx = A.$$

In general, the area under the hyperbola over the interval $[2^n, 2^{n+1}]$ is

$$\int_{2^n}^{2^{n+1}} \frac{1}{x}\,dx = \int_1^{2^{n+1}/2^n} \frac{1}{x}\,dx = \int_1^2 \frac{1}{x}\,dx = A.$$

102. Show that the two regions in Figure 6 have the same area. Then use the identity $\cos^2 u = \frac{1}{2}(1 + \cos 2u)$ to compute the second area.

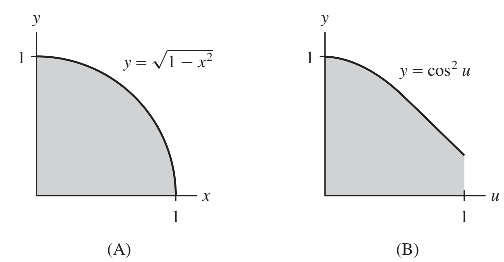


FIGURE 6

SOLUTION The area of the region in Figure 6(A) is given by $\int_0^1 \sqrt{1 - x^2}\,dx$. Let $x = \sin u$. Then $dx = \cos u\,du$ and $\sqrt{1 - x^2} = \sqrt{1 - \sin^2 u} = \cos u$. Hence,

$$\int_0^1 \sqrt{1 - x^2}\,dx = \int_0^{\pi/2} \cos u \cdot \cos u\,du = \int_0^{\pi/2} \cos^2 u\,du.$$

This last integral represents the area of the region in Figure 6(B). The two regions in Figure 6 therefore have the same area.

Let's now focus on the definite integral $\int_0^{\pi/2} \cos^2 u\,du$. Using the trigonometric identity $\cos^2 u = \frac{1}{2}(1 + \cos 2u)$, we have

$$\int_0^{\pi/2} \cos^2 u\,du = \frac{1}{2}\int_0^{\pi/2} 1 + \cos 2u\,du = \frac{1}{2}\left(u + \frac{1}{2}\sin 2u\right)\Big|_0^{\pi/2} = \frac{1}{2} \cdot \frac{\pi}{2} - 0 = \frac{\pi}{4}.$$

103. Area of an Ellipse Prove the formula $A = \pi ab$ for the area of the ellipse with equation (Figure 7)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Hint: Use a change of variables to show that A is equal to ab times the area of the unit circle.

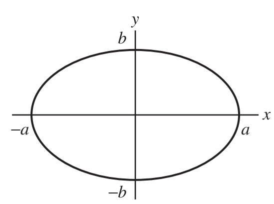


FIGURE 7 Graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

SOLUTION Consider the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; here $a, b > 0$. The area between the part of the ellipse in the upper half-plane, $y = f(x) = \sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)}$, and the x-axis is $\int_{-a}^{a} f(x)\,dx$. By symmetry, the part of the elliptical region in the lower half-plane has the same area. Accordingly, the area enclosed by the ellipse is

$$2\int_{-a}^{a} f(x)\,dx = 2\int_{-a}^{a} \sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)}\,dx = 2b\int_{-a}^{a} \sqrt{1 - (x/a)^2}\,dx$$

Now, let $u = x/a$. Then $x = au$ and $a\,du = dx$. Accordingly,

$$2b\int_{-a}^{a} \sqrt{1 - \left(\frac{x}{a}\right)^2}\,dx = 2ab\int_{-1}^{1} \sqrt{1 - u^2}\,du = 2ab\left(\frac{\pi}{2}\right) = \pi ab$$

Here we recognized that $\int_{-1}^{1} \sqrt{1 - u^2}\,du$ represents the area of the upper unit semicircular disk, which by Exercise 102 is $2(\frac{\pi}{4}) = \frac{\pi}{2}$.

5.7 Further Transcendental Functions

Preliminary Questions

1. Find b such that $\int_1^b \frac{dx}{x}$ is equal to

(a) $\ln 3$ **(b)** 3

SOLUTION For $b > 0$,

$$\int_1^b \frac{dx}{x} = \ln|x|\Big|_1^b = \ln b - \ln 1 = \ln b.$$

(a) For the value of the definite integral to equal $\ln 3$, we must have $b = 3$.
(b) For the value of the definite integral to equal 3, we must have $b = e^3$.

2. Find b such that $\int_0^b \frac{dx}{1 + x^2} = \frac{\pi}{3}$.

SOLUTION In general,

$$\int_0^b \frac{dx}{1 + x^2} = \tan^{-1} x\Big|_0^b = \tan^{-1} b - \tan^{-1} 0 = \tan^{-1} b.$$

For the value of the definite integral to equal $\frac{\pi}{3}$, we must have

$$\tan^{-1} b = \frac{\pi}{3} \quad \text{or} \quad b = \tan\frac{\pi}{3} = \sqrt{3}.$$

3. Which integral should be evaluated using substitution?

(a) $\int \frac{9\,dx}{1 + x^2}$ **(b)** $\int \frac{dx}{1 + 9x^2}$

SOLUTION Use the substitution $u = 3x$ on the integral in **(b)**.

4. Which relation between x and u yields $\sqrt{16 + x^2} = 4\sqrt{1 + u^2}$?

SOLUTION To transform $\sqrt{16 + x^2}$ into $4\sqrt{1 + u^2}$, make the substitution $x = 4u$.

Exercises

In Exercises 1–10, evaluate the definite integral.

1. $\int_1^9 \frac{dx}{x}$

SOLUTION $\int_1^9 \frac{1}{x}\,dx = \ln|x|\Big|_1^9 = \ln 9 - \ln 1 = \ln 9.$

2. $\int_4^{20} \frac{dx}{x}$

SOLUTION $\int_4^{20} \frac{1}{x}\,dx = \ln|x|\Big|_4^{20} = \ln 20 - \ln 4 = \ln 5.$

3. $\displaystyle\int_1^{e^3} \frac{1}{t}\,dt$

SOLUTION $\displaystyle\int_1^{e^3} \frac{1}{t}\,dt = \ln|t|\Big|_1^{e^3} = \ln e^3 - \ln 1 = 3.$

4. $\displaystyle\int_{-e^2}^{-e} \frac{1}{t}\,dt$

SOLUTION $\displaystyle\int_{-e^2}^{-e} \frac{1}{t}\,dt = \ln|t|\Big|_{-e^2}^{-e} = \ln|-e| - \ln|-e^2| = \ln\frac{e}{e^2} = \ln(1/e) = -1.$

5. $\displaystyle\int_2^{12} \frac{dt}{3t+4}$

SOLUTION Let $u = 3t+4$. Then $du = 3\,dt$ and

$$\int_2^{12} \frac{dt}{3t+4} = \frac{1}{3}\int_{10}^{40} \frac{du}{u} = \frac{1}{3}\ln|u|\Big|_{10}^{40} = \frac{1}{3}(\ln 40 - \ln 10) = \frac{1}{3}\ln 4.$$

6. $\displaystyle\int_e^{e^3} \frac{dt}{t\ln t}$

SOLUTION Let $u = \ln t$. Then $du = (1/t)dt$ and

$$\int_e^{e^3} \frac{1}{t\ln t}\,dt = \int_1^3 \frac{du}{u} = \ln|u|\Big|_1^3 = \ln 3 - \ln 1 = \ln 3.$$

7. $\displaystyle\int_{\tan 1}^{\tan 8} \frac{dx}{x^2+1}$

SOLUTION $\displaystyle\int_{\tan 1}^{\tan 8} \frac{dx}{1+x^2} = \tan^{-1} x\Big|_{\tan 1}^{\tan 8} = \tan^{-1}(\tan 8) - \tan^{-1}(\tan 1) = 8 - 1 = 7.$

8. $\displaystyle\int_2^{7} \frac{x\,dx}{x^2+1}$

SOLUTION Let $u = x^2+1$. Then $du = 2x\,dx$ and

$$\int_2^7 \frac{x\,dx}{x^2+1} = \frac{1}{2}\int_5^{50} \frac{du}{u} = \frac{1}{2}\ln|u|\Big|_5^{50} = \frac{1}{2}(\ln 50 - \ln 5) = \frac{1}{2}\ln 10.$$

9. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$

SOLUTION $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big|_0^{1/2} = \sin^{-1}\frac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6}.$

10. $\displaystyle\int_{-2}^{-2/\sqrt{3}} \frac{dx}{|x|\sqrt{x^2-1}}$

SOLUTION $\displaystyle\int_{-2}^{-2/\sqrt{3}} \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x\Big|_{-2}^{-2/\sqrt{3}} = \sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) - \sec^{-1}(-2) = \frac{5\pi}{6} - \frac{2\pi}{3} = \frac{\pi}{6}.$

11. Use the substitution $u = x/3$ to prove

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\frac{x}{3} + C$$

SOLUTION Let $u = x/3$. Then, $x = 3u$, $dx = 3\,du$, $9 + x^2 = 9(1+u^2)$, and

$$\int \frac{dx}{9+x^2} = \int \frac{3\,du}{9(1+u^2)} = \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1} u + C = \frac{1}{3}\tan^{-1}\frac{x}{3} + C.$$

12. Use the substitution $u = 2x$ to evaluate $\displaystyle\int \frac{dx}{4x^2+1}$.

SOLUTION Let $u = 2x$. Then, $x = u/2$, $dx = \frac{1}{2}\,du$, $4x^2+1 = u^2+1$, and

$$\int \frac{dx}{4x^2+1} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1} 2x + C.$$

In Exercises 13–32, calculate the integral.

13. $\int_0^3 \frac{dx}{x^2+3}$

SOLUTION Let $x = \sqrt{3}u$. Then $dx = \sqrt{3}\,du$ and

$$\int_0^3 \frac{dx}{x^2+3} = \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} \frac{du}{u^2+1} = \frac{1}{\sqrt{3}}\tan^{-1} u\Big|_0^{\sqrt{3}} = \frac{1}{\sqrt{3}}(\tan^{-1}\sqrt{3} - \tan^{-1} 0) = \frac{\pi}{3\sqrt{3}}.$$

14. $\int_0^4 \frac{dt}{4t^2+9}$

SOLUTION Let $t = (3/2)u$. Then $dt = (3/2)\,du$, $4t^2 + 9 = 9t^2 + 9 = 9(t^2+1)$, and

$$\int_0^4 \frac{dt}{4t^2+9} = \frac{1}{6}\int_0^{8/3} \frac{du}{u^2+1} = \frac{1}{6}\tan^{-1} u\Big|_0^{8/3} = \frac{1}{6}\tan^{-1}\frac{8}{3}.$$

15. $\int \frac{dt}{\sqrt{1-16t^2}}$

SOLUTION Let $u = 4t$. Then $du = 4\,dt$, and

$$\int \frac{dt}{\sqrt{1-16t^2}} = \int \frac{du}{4\sqrt{1-u^2}} = \frac{1}{4}\sin^{-1} u + C = \frac{1}{4}\sin^{-1}(4t) + C.$$

16. $\int_{-1/5}^{1/5} \frac{dx}{\sqrt{4-25x^2}}$

SOLUTION Let $x = 2u/5$. Then

$$dx = \frac{2}{5}\,du, \quad 4 - 25x^2 = 4(1-u^2),$$

and

$$\begin{aligned}\int_{-1/5}^{1/5} \frac{dx}{\sqrt{4-25x^2}} &= \frac{2}{5}\int_{-1/2}^{1/2} \frac{1}{\sqrt{4(1-u^2)}}\,du \\ &= \frac{1}{5}\sin^{-1} u\Big|_{-1/2}^{1/2} \\ &= \frac{1}{5}\left(\sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right) = \frac{\pi}{15}.\end{aligned}$$

17. $\int \frac{dt}{\sqrt{5-3t^2}}$

SOLUTION Let $t = \sqrt{5/3}u$. Then $dt = \sqrt{5/3}\,du$ and

$$\int \frac{dt}{\sqrt{5-3t^2}} = \int \frac{\sqrt{5/3}\,du}{\sqrt{5}\sqrt{1-t^2}} = \frac{1}{\sqrt{3}}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{\sqrt{3}}\sin^{-1} u + C = \frac{1}{\sqrt{3}}\sin^{-1}\sqrt{\frac{3}{5}}t + C.$$

18. $\int_{1/2\sqrt{2}}^{1/2} \frac{dx}{x\sqrt{16x^2-1}}$

SOLUTION Let $x = u/4$. Then $dx = du/4$, $16x^2 - 1 = u^2 - 1$ and

$$\int_{1/2\sqrt{2}}^{1/2} \frac{dx}{x\sqrt{16x^2-1}} = \int_{\sqrt{2}}^{2} \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u\Big|_{\sqrt{2}}^{2} = \sec^{-1} 2 - \sec^{-1}\sqrt{2} = \frac{\pi}{12}.$$

19. $\int \frac{dx}{x\sqrt{12x^2-3}}$

SOLUTION Let $u = 2x$. Then $du = 2\,dx$ and

$$\int \frac{dx}{x\sqrt{12x^2-3}} = \frac{1}{\sqrt{3}}\int \frac{du}{u\sqrt{u^2-1}} = \frac{1}{\sqrt{3}}\sec^{-1} u + C = \frac{1}{\sqrt{3}}\sec^{-1}(2x) + C.$$

20. $\displaystyle\int \frac{x\,dx}{x^4+1}$

SOLUTION Let $u = x^2$. Then $du = 2x\,dx$ and

$$\int \frac{x\,dx}{x^4+1} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1} x^2 + C.$$

21. $\displaystyle\int \frac{dx}{x\sqrt{x^4-1}}$

SOLUTION Let $u = x^2$. Then $du = 2x\,dx$, and

$$\int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{du}{2u\sqrt{u^2-1}} = \frac{1}{2}\sec^{-1} u + C = \frac{1}{2}\sec^{-1} x^2 + C.$$

22. $\displaystyle\int_{-1/2}^{0} \frac{(x+1)\,dx}{\sqrt{1-x^2}}$

SOLUTION Observe that

$$\int \frac{(x+1)\,dx}{\sqrt{1-x^2}} = \int \frac{x\,dx}{\sqrt{1-x^2}} + \int \frac{dx}{\sqrt{1-x^2}}.$$

In the first integral on the right, we let $u = 1 - x^2$, $du = -2x\,dx$. Thus

$$\int \frac{(x+1)\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{du}{u^{1/2}} + \int \frac{1\,dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + \sin^{-1} x + C.$$

Finally,

$$\int_{-1/2}^{0} \frac{(x+1)\,dx}{\sqrt{1-x^2}} = \left(-\sqrt{1-x^2} + \sin^{-1} x\right)\Big|_{-1/2}^{0} = -1 + \frac{\sqrt{3}}{2} + \frac{\pi}{6}.$$

23. $\displaystyle\int_{-\ln 2}^{0} \frac{e^x\,dx}{1+e^{2x}}$

SOLUTION Let $u = e^x$. Then $du = e^x\,dx$, and

$$\int_{-\ln 2}^{0} \frac{e^x\,dx}{1+e^{2x}} = \int_{1/2}^{1} \frac{du}{1+u^2} = \tan^{-1} u\Big|_{1/2}^{1} = \frac{\pi}{4} - \tan^{-1}(1/2).$$

24. $\displaystyle\int \frac{\ln(\cos^{-1} x)\,dx}{(\cos^{-1} x)\sqrt{1-x^2}}$

SOLUTION Let $u = \ln\cos^{-1} x$. Then $du = \dfrac{1}{\cos^{-1} x} \cdot \dfrac{-1}{\sqrt{1-x^2}}$, and

$$\int \frac{\ln(\cos^{-1} x)\,dx}{(\cos^{-1} x)\sqrt{1-x^2}} = -\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}(\ln\cos^{-1} x)^2 + C.$$

25. $\displaystyle\int \frac{\tan^{-1} x\,dx}{1+x^2}$

SOLUTION Let $u = \tan^{-1} x$. Then $du = \dfrac{dx}{1+x^2}$, and

$$\int \frac{\tan^{-1} x\,dx}{1+x^2} = \int u\,du = \frac{1}{2}u^2 + C = \frac{(\tan^{-1} x)^2}{2} + C.$$

26. $\displaystyle\int_{1}^{\sqrt{3}} \frac{dx}{(\tan^{-1} x)(1+x^2)}$

SOLUTION Let $u = \tan^{-1} x$. Then $du = \dfrac{dx}{1+x^2}$, and

$$\int_{1}^{\sqrt{3}} \frac{dx}{(\tan^{-1} x)(1+x^2)} = \int_{\pi/4}^{\pi/3} \frac{1}{u}\,du = \ln|u|\Big|_{\pi/4}^{\pi/3} = \ln\frac{\pi}{3} - \ln\frac{\pi}{4} = \ln\frac{4}{3}.$$

27. $\int_0^1 3^x\,dx$

SOLUTION $\int_0^1 3^x\,dx = \frac{3^x}{\ln 3}\bigg|_0^1 = \frac{1}{\ln 3}(3-1) = \frac{2}{\ln 3}.$

28. $\int_0^1 3^{-x}\,dx$

SOLUTION Let $u = -x$. Then $du = -dx$ and

$$\int_0^1 3^{-x}\,dx = -\int_0^{-1} 3^u\,du = -\frac{3^u}{\ln 3}\bigg|_0^{-1} = \frac{1}{\ln 3}\left(-\frac{1}{3}+1\right) = \frac{2}{3\ln 3}.$$

29. $\int_0^{\log_4(3)} 4^x\,dx$

SOLUTION $\int_0^{\log_4(3)} 4^x\,dx = \frac{4^x}{\ln 4}\bigg|_0^{\log_4 3} = \frac{1}{\ln 4}(3-1) = \frac{2}{\ln 4} = \frac{1}{\ln 2}.$

30. $\int_0^1 t5^{t^2}\,dt$

SOLUTION Let $u = t^2$. Then $du = 2t\,dt$ and

$$\int_0^1 t5^{t^2}\,dt = \frac{1}{2}\int_0^1 5^u\,du = \frac{5^u}{2\ln 5}\bigg|_0^1 = \frac{5}{2\ln 5} - \frac{1}{2\ln 5} = \frac{2}{\ln 5}.$$

31. $\int 9^x \sin(9^x)\,dx$

SOLUTION Let $u = 9^x$. Then $du = 9^x \ln 9\,dx$ and

$$\int 9^x \sin(9^x)\,dx = \frac{1}{\ln 9}\int \sin u\,du = -\frac{1}{\ln 9}\cos u + C = -\frac{1}{\ln 9}\cos(9^x) + C.$$

32. $\int \frac{dx}{\sqrt{5^{2x}-1}}$

SOLUTION First, rewrite

$$\int \frac{dx}{\sqrt{5^{2x}-1}} = \int \frac{dx}{5^x\sqrt{1-5^{-2x}}} = \int \frac{5^{-x}\,dx}{\sqrt{1-5^{-2x}}}.$$

Now, let $u = 5^{-x}$. Then $du = -5^{-x}\ln 5\,dx$ and

$$\int \frac{dx}{\sqrt{5^{2x}-1}} = -\frac{1}{\ln 5}\int \frac{du}{\sqrt{1-u^2}} = -\frac{1}{\ln 5}\sin^{-1} u + C = -\frac{1}{\ln 5}\sin^{-1}(5^{-x}) + C.$$

In Exercises 33–70, evaluate the integral using the methods covered in the text so far.

33. $\int ye^{y^2}\,dy$

SOLUTION Use the substitution $u = y^2$, $du = 2y\,dy$. Then

$$\int ye^{y^2}\,dy = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{y^2} + C.$$

34. $\int \frac{dx}{3x+5}$

SOLUTION Let $u = 3x+5$. Then $du = 3\,dx$ and

$$\int \frac{dx}{3x+5} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|3x+5| + C.$$

35. $\int \frac{x\,dx}{\sqrt{4x^2+9}}$

SOLUTION Let $u = 4x^2+9$. Then $du = 8x\,dx$ and

$$\int \frac{x}{\sqrt{4x^2+9}}\,dx = \frac{1}{8}\int u^{-1/2}\,du = \frac{1}{4}u^{1/2} + C = \frac{1}{4}\sqrt{4x^2+9} + C.$$

36. $\int (x - x^{-2})^2\,dx$

SOLUTION $\int (x - x^{-2})^2\,dx = \int (x^2 - 2x^{-1} + x^{-4})\,dx = \frac{1}{3}x^3 - 2\ln|x| - \frac{1}{3}x^{-3} + C.$

37. $\int 7^{-x}\,dx$

SOLUTION Let $u = -x$. Then $du = -dx$ and

$$\int 7^{-x}\,dx = -\int 7^u\,du = -\frac{7^u}{\ln 7} + C = -\frac{7^{-x}}{\ln 7} + C.$$

38. $\int e^{9-12t}\,dt$

SOLUTION Let $u = 9 - 12t$. Then $du = -12\,dt$ and

$$\int e^{9-12t}\,dt = -\frac{1}{12}\int e^u\,du = -\frac{1}{12}e^u + C = -\frac{1}{12}e^{9-12t} + C.$$

39. $\int \sec^2\theta\tan^7\theta\,d\theta$

SOLUTION Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$ and

$$\int \sec^2\theta\tan^7\theta\,d\theta = \int u^7\,du = \frac{1}{8}u^8 + C = \frac{1}{8}\tan^8\theta + C.$$

40. $\int \frac{\cos(\ln t)\,dt}{t}$

SOLUTION Let $u = \ln t$. Then $du = dt/t$ and

$$\int \frac{\cos(\ln t)\,dt}{t} = \int \cos u\,du = \sin u + C = \sin(\ln t) + C.$$

41. $\int \frac{t\,dt}{\sqrt{7 - t^2}}$

SOLUTION Let $u = 7 - t^2$. Then $du = -2t\,dt$ and

$$\int \frac{t\,dt}{\sqrt{7 - t^2}} = -\frac{1}{2}\int u^{-1/2}\,du = -u^{1/2} + C = -\sqrt{7 - t^2} + C.$$

42. $\int 2^x e^{4x}\,dx$

SOLUTION First, note that

$$2^x = e^{x\ln 2} \quad \text{so} \quad 2^x e^{4x} = e^{(4+\ln 2)x}.$$

Thus,

$$\int 2^x e^{4x}\,dx = \int e^{(4+\ln 2)x}\,dx = \frac{1}{4 + \ln 2}e^{(4+\ln 2)x} + C.$$

43. $\int \frac{(3x + 2)\,dx}{x^2 + 4}$

SOLUTION Write

$$\int \frac{(3x + 2)\,dx}{x^2 + 4} = \int \frac{3x\,dx}{x^2 + 4} + \int \frac{2\,dx}{x^2 + 4}.$$

In the first integral, let $u = x^2 + 4$. Then $du = 2x\,dx$ and

$$\int \frac{3x\,dx}{x^2 + 4} = \frac{3}{2}\int \frac{du}{u} - \frac{3}{2}\ln|u| + C_1 = \frac{3}{2}\ln(x^2 + 4) + C_1.$$

For the second integral, let $x = 2u$. Then $dx = 2\,du$ and

$$\int \frac{2\,dx}{x^2 + 4} = \int \frac{du}{u^2 + 1} = \tan^{-1} u + C_2 = \tan^{-1}(x/2) + C_2.$$

Combining these two results yields

$$\int \frac{(3x+2)\,dx}{x^2+4} = \frac{3}{2}\ln(x^2+4) + \tan^{-1}(x/2) + C.$$

44. $\int \tan(4x+1)\,dx$

SOLUTION First we rewrite $\int \tan(4x+1)\,dx$ as $\int \frac{\sin(4x+1)}{\cos(4x+1)}\,dx$. Let $u = \cos(4x+1)$. Then $du = -4\sin(4x+1)\,dx$, and

$$\int \frac{\sin(4x+1)}{\cos(4x+1)}dx = -\frac{1}{4}\int \frac{du}{u} = -\frac{1}{4}\ln|\cos(4x+1)| + C.$$

45. $\int \frac{dx}{\sqrt{1-16x^2}}$

SOLUTION Let $u = 4x$. Then $du = 4\,dx$ and

$$\int \frac{dx}{\sqrt{1-16x^2}} = \frac{1}{4}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{4}\sin^{-1}u + C = \frac{1}{4}\sin^{-1}(4x) + C.$$

46. $\int e^t\sqrt{e^t+1}\,dt$

SOLUTION Use the substitution $u = e^t + 1$, $du = e^t\,dt$. Then

$$\int e^t\sqrt{e^t+1}\,dt = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(e^t+1)^{3/2} + C.$$

47. $\int (e^{-x} - 4x)\,dx$

SOLUTION First, observe that

$$\int (e^{-x} - 4x)\,dx = \int e^{-x}\,dx - \int 4x\,dx = \int e^{-x}\,dx - 2x^2.$$

In the remaining integral, use the substitution $u = -x$, $du = -dx$. Then

$$\int e^{-x}\,dx = -\int e^u\,du = -e^u + C = -e^{-x} + C.$$

Finally,

$$\int (e^{-x} - 4x)\,dx = -e^{-x} - 2x^2 + C.$$

48. $\int (7 - e^{10x})\,dx$

SOLUTION First, observe that

$$\int (7 - e^{10x})\,dx = \int 7\,dx - \int e^{10x}\,dx = 7x - \int e^{10x}\,dx.$$

In the remaining integral, use the substitution $u = 10x$, $du = 10\,dx$. Then

$$\int e^{10x}\,dx = \frac{1}{10}\int e^u\,du = \frac{1}{10}e^u + C = \frac{1}{10}e^{10x} + C.$$

Finally,

$$\int (7 - e^{10x})\,dx = 7x - \frac{1}{10}e^{10x} + C.$$

49. $\int \frac{e^{2x} - e^{4x}}{e^x}\,dx$

SOLUTION

$$\int \left(\frac{e^{2x} - e^{4x}}{e^x}\right)dx = \int (e^x - e^{3x})\,dx = e^x - \frac{e^{3x}}{3} + C.$$

50. $\displaystyle\int \frac{dx}{x\sqrt{25x^2-1}}$

SOLUTION Let $u = 5x$. Then $du = 5\,dx$ and

$$\int \frac{dx}{x\sqrt{25x^2-1}} = \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + C = \sec^{-1}(5x) + C.$$

51. $\displaystyle\int \frac{(x+5)\,dx}{\sqrt{4-x^2}}$

SOLUTION Write

$$\int \frac{(x+5)\,dx}{\sqrt{4-x^2}} = \int \frac{x\,dx}{\sqrt{4-x^2}} + \int \frac{5\,dx}{\sqrt{4-x^2}}.$$

In the first integral, let $u = 4 - x^2$. Then $du = -2x\,dx$ and

$$\int \frac{x\,dx}{\sqrt{4-x^2}} = -\frac{1}{2}\int u^{-1/2}\,du = -u^{1/2} + C_1 = -\sqrt{4-x^2} + C_1.$$

In the second integral, let $x = 2u$. Then $dx = 2\,du$ and

$$\int \frac{5\,dx}{\sqrt{4-x^2}} = 5\int \frac{du}{\sqrt{1-u^2}} = 5\sin^{-1} u + C_2 = 5\sin^{-1}(x/2) + C_2.$$

Combining these two results yields

$$\int \frac{(x+5)\,dx}{\sqrt{4-x^2}} = -\sqrt{4-x^2} + 5\sin^{-1}(x/2) + C.$$

52. $\displaystyle\int (t+1)\sqrt{t+1}\,dt$

SOLUTION Let $u = t + 1$. Then $du = dt$ and

$$\int (t+1)\sqrt{t+1}\,dt = \int u^{3/2}\,du = \frac{2}{5}u^{5/2} + C = \frac{2}{5}(t+1)^{5/2} + C.$$

53. $\displaystyle\int e^x \cos(e^x)\,dx$

SOLUTION Use the substitution $u = e^x$, $du = e^x\,dx$. Then

$$\int e^x \cos(e^x)\,dx = \int \cos u\,du = \sin u + C = \sin(e^x) + C.$$

54. $\displaystyle\int \frac{e^x}{\sqrt{e^x+1}}\,dx$

SOLUTION Use the substitution $u = e^x + 1$, $du = e^x\,dx$. Then

$$\int \frac{e^x}{\sqrt{e^x+1}}\,dx = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{e^x+1} + C.$$

55. $\displaystyle\int \frac{dx}{\sqrt{9-16x^2}}$

SOLUTION First rewrite

$$\int \frac{dx}{\sqrt{9-16x^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{1-\left(\frac{4}{3}x\right)^2}}.$$

Now, let $u = \frac{4}{3}x$. Then $du = \frac{4}{3}\,dx$ and

$$\int \frac{dx}{\sqrt{9-16x^2}} = \frac{1}{4}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{4}\sin^{-1} u + C = \frac{1}{4}\sin^{-1}\left(\frac{4x}{3}\right) + C.$$

56. $\int \frac{dx}{(4x-1)\ln(8x-2)}$

SOLUTION Let $u = \ln(8x-2)$. Then $du = \frac{8}{8x-2}dx = \frac{4}{4x-1}dx$, and

$$\int \frac{dx}{(4x-1)\ln(8x-2)} = \frac{1}{4}\int \frac{du}{u} = \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln|\ln(8x-2)| + C.$$

57. $\int e^x(e^{2x}+1)^3\,dx$

SOLUTION Use the substitution $u = e^x$, $du = e^x\,dx$. Then

$$\begin{aligned}\int e^x(e^{2x}+1)^3\,dx &= \int \left(u^2+1\right)^3\,du = \int \left(u^6+3u^4+3u^2+1\right)\,du\\ &= \frac{1}{7}u^7 + \frac{3}{5}u^5 + u^3 + u + C = \frac{1}{7}(e^x)^7 + \frac{3}{5}(e^x)^5 + (e^x)^3 + e^x + C\\ &= \frac{e^{7x}}{7} + \frac{3e^{5x}}{5} + e^{3x} + e^x + C.\end{aligned}$$

58. $\int \frac{dx}{x(\ln x)^5}$

SOLUTION Let $u = \ln x$. Then $du = dx/x$ and

$$\int \frac{dx}{x(\ln x)^5} = \int u^{-5}\,du = -\frac{1}{4}u^{-4} + C = -\frac{1}{4(\ln x)^4} + C.$$

59. $\int \frac{x^2\,dx}{x^3+2}$

SOLUTION Let $u = x^3+2$. Then $du = 3x^2\,dx$, and

$$\int \frac{x^2\,dx}{x^3+2} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|x^3+2| + C.$$

60. $\int \frac{(3x-1)\,dx}{9-2x+3x^2}$

SOLUTION Let $u = 9-2x+3x^2$. Then $du = (-2+6x)\,dx = 2(3x-1)\,dx$, and

$$\int \frac{(3x-1)dx}{9-2x+3x^2} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(9-2x+3x^2) + C.$$

61. $\int \cot x\,dx$

SOLUTION We rewrite $\int \cot x\,dx$ as $\int \frac{\cos x}{\sin x}\,dx$. Let $u = \sin x$. Then $du = \cos x\,dx$, and

$$\int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|\sin x| + C.$$

62. $\int \frac{\cos x}{2\sin x+3}\,dx$

SOLUTION Let $u = 2\sin x + 3$. Then $du = 2\cos x\,dx$, and

$$\int \frac{\cos x}{2\sin x+3}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(2\sin x+3) + C,$$

where we have used the fact that $2\sin x + 3 \geq 1$ to drop the absolute value.

63. $\int \frac{4\ln x+5}{x}\,dx$

SOLUTION Let $u = 4\ln x + 5$. Then $du = (4/x)dx$, and

$$\int \frac{4\ln x+5}{x}dx = \frac{1}{4}\int u\,du = \frac{1}{8}u^2 + C = \frac{1}{8}(4\ln x+5)^2 + C.$$

64. $\int (\sec\theta\tan\theta)5^{\sec\theta}\, d\theta$

SOLUTION Let $u = \sec\theta$. Then $du = \sec\theta\tan\theta\, d\theta$ and

$$\int (\sec\theta\tan\theta)5^{\sec\theta}\, d\theta = \int 5^u\, du = \frac{5^u}{\ln 5} + C = \frac{5^{\sec\theta}}{\ln 5} + C.$$

65. $\int x3^{x^2}\, dx$

SOLUTION Let $u = x^2$. Then $du = 2x\, dx$, and

$$\int x3^{x^2}\, dx = \frac{1}{2}\int 3^u du = \frac{1}{2}\frac{3^u}{\ln 3} + C = \frac{3^{x^2}}{2\ln 3} + C.$$

66. $\int \frac{\ln(\ln x)}{x\ln x}\, dx$

SOLUTION Let $u = \ln(\ln x)$. Then $du = \frac{1}{\ln x}\cdot\frac{1}{x}\, dx$ and

$$\int \frac{\ln(\ln x)}{x\ln x}\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{(\ln(\ln x))^2}{2} + C.$$

67. $\int \cot x\ln(\sin x)\, dx$

SOLUTION Let $u = \ln(\sin x)$. Then

$$du = \frac{1}{\sin x}\cdot\cos x\, dx = \cot x\, dx,$$

and

$$\int \cot x\ln(\sin x)\, dx = \int u\, du = \frac{u^2}{2} + C = \frac{(\ln(\sin x))^2}{2} + C.$$

68. $\int \frac{t\, dt}{\sqrt{1 - t^4}}$

SOLUTION Let $u = t^2$. Then $du = 2t\, dt$ and

$$\int \frac{t\, dt}{\sqrt{1 - t^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1 - u^2}} = \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1} t^2 + C.$$

69. $\int t^2\sqrt{t - 3}\, dt$

SOLUTION Let $u = t - 3$. Then $t = u + 3$, $du = dt$ and

$$\begin{aligned}\int t^2\sqrt{t - 3}\, dt &= \int (u + 3)^2\sqrt{u}\, du \\ &= \int (u^2 + 6u + 9)\sqrt{u}\, du = \int (u^{5/2} + 6u^{3/2} + 9u^{1/2})\, du \\ &= \frac{2}{7}u^{7/2} + \frac{12}{5}u^{5/2} + 6u^{3/2} + C \\ &= \frac{2}{7}(t - 3)^{7/2} + \frac{12}{5}(t - 3)^{5/2} + 6(t - 3)^{3/2} + C.\end{aligned}$$

70. $\int \cos x 5^{-2\sin x}\, dx$

SOLUTION Let $u = -2\sin x$. Then $du = -2\cos x\, dx$ and

$$\int \cos x 5^{-2\sin x}\, dx = -\frac{1}{2}\int 5^u\, du = -\frac{5^u}{2ln5} + C = -\frac{5^{-2\sin x}}{2\ln 5} + C.$$

71. Use Figure 4 to prove

$$\int_0^x \sqrt{1-t^2}\,dt = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x$$

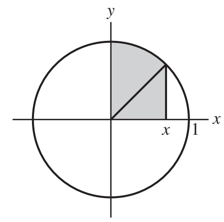


FIGURE 4

SOLUTION The definite integral $\int_0^x \sqrt{1-t^2}\,dt$ represents the area of the region under the upper half of the unit circle from 0 to x. The region consists of a sector of the circle and a right triangle. The sector has a central angle of $\frac{\pi}{2} - \theta$, where $\cos\theta = x$. Hence, the sector has an area of

$$\frac{1}{2}(1)^2\left(\frac{\pi}{2} - \cos^{-1} x\right) = \frac{1}{2}\sin^{-1} x.$$

The right triangle has a base of length x, a height of $\sqrt{1-x^2}$, and hence an area of $\frac{1}{2}x\sqrt{1-x^2}$. Thus,

$$\int_0^x \sqrt{1-t^2}\,dt = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x.$$

72. Use the substitution $u = \tan x$ to evaluate

$$\int \frac{dx}{1+\sin^2 x}.$$

Hint: Show that

$$\frac{dx}{1+\sin^2 x} = \frac{du}{1+2u^2}$$

SOLUTION If $u = \tan x$, then $du = \sec^2 x\,dx$ and

$$\frac{du}{1+2u^2} = \frac{\sec^2 x\,dx}{1+2\tan^2 x} = \frac{dx}{\cos^2 x + 2\sin^2 x} = \frac{dx}{\cos^2 x + \sin^2 x + \sin^2 x} = \frac{dx}{1+\sin^2 x}.$$

Thus

$$\int \frac{dx}{1+\sin^2 x} = \int \frac{du}{1+2u^2} = \int \frac{du}{1+(\sqrt{2}u)^2} = \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}u) + C = \frac{1}{\sqrt{2}}\tan^{-1}((\tan x)\sqrt{2}) + C.$$

73. Prove:

$$\int \sin^{-1} t\,dt = \sqrt{1-t^2} + t\sin^{-1} t.$$

SOLUTION Let $G(t) = \sqrt{1-t^2} + t\sin^{-1} t$. Then

$$G'(t) = \frac{d}{dt}\sqrt{1-t^2} + \frac{d}{dt}\left(t\sin^{-1} t\right) = \frac{-t}{\sqrt{1-t^2}} + \left(t \cdot \frac{d}{dt}\sin^{-1} t + \sin^{-1} t\right)$$
$$= \frac{-t}{\sqrt{1-t^2}} + \left(\frac{t}{\sqrt{1-t^2}} + \sin^{-1} t\right) = \sin^{-1} t.$$

This proves the formula $\int \sin^{-1} t\,dt = \sqrt{1-t^2} + t\sin^{-1} t$.

74. **(a)** Verify for $r \neq 0$:

$$\int_0^T te^{rt}\,dt = \frac{e^{rT}(rT-1)+1}{r^2} \qquad \boxed{6}$$

Hint: For fixed r, let $F(T)$ be the value of the integral on the left. By FTC II, $F'(t) = te^{rt}$ and $F(0) = 0$. Show that the same is true of the function on the right.

(b) Use L'Hôpital's Rule to show that for fixed T, the limit as $r \to 0$ of the right-hand side of Eq. (6) is equal to the value of the integral for $r = 0$.

SOLUTION

(a) Let

$$f(t) = \frac{e^{rt}}{r^2}(rt - 1) + \frac{1}{r^2}.$$

Then

$$f'(t) = \frac{1}{r^2}\left(e^{rt}r + (rt - 1)(re^{rt})\right) = te^{rt}$$

and

$$f(0) = -\frac{1}{r^2} + \frac{1}{r^2} = 0,$$

as required.

(b) Using L'Hôpital's Rule,

$$\lim_{r\to 0} \frac{e^{rT}(rT - 1) + 1}{r^2} = \lim_{r\to 0} \frac{Te^{rT} + (rT - 1)(Te^{rT})}{2r} = \lim_{r\to 0} \frac{rT^2e^{rT}}{2r} = \lim_{r\to 0} \frac{T^2e^{rT}}{2} = \frac{T^2}{2}.$$

If $r = 0$ then, $\int_0^T te^{rt}\,dt = \int_0^T t\,dt = \left.\frac{t^2}{2}\right|_0^T = \frac{T^2}{2}$.

Further Insights and Challenges

75. Recall that if $f(t) \ge g(t)$ for $t \ge 0$, then for all $x \ge 0$,

$$\int_0^x f(t)\,dt \ge \int_0^x g(t)\,dt \qquad \boxed{7}$$

The inequality $e^t \ge 1$ holds for $t \ge 0$ because $e > 1$. Use Eq. (7) to prove that $e^x \ge 1 + x$ for $x \ge 0$. Then prove, by successive integration, the following inequalities (for $x \ge 0$):

$$e^x \ge 1 + x + \frac{1}{2}x^2, \qquad e^x \ge 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

SOLUTION Integrating both sides of the inequality $e^t \ge 1$ yields

$$\int_0^x e^t\,dt = e^x - 1 \ge x \quad \text{or} \quad e^x \ge 1 + x.$$

Integrating both sides of this new inequality then gives

$$\int_0^x e^t\,dt = e^x - 1 \ge x + x^2/2 \quad \text{or} \quad e^x \ge 1 + x + x^2/2.$$

Finally, integrating both sides again gives

$$\int_0^x e^t\,dt = e^x - 1 \ge x + x^2/2 + x^3/6 \quad \text{or} \quad e^x \ge 1 + x + x^2/2 + x^3/6$$

as requested.

76. Generalize Exercise 75; that is, use induction (if you are familiar with this method of proof) to prove that for all $n \ge 0$,

$$e^x \ge 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{n!}x^n \quad (x \ge 0)$$

SOLUTION For $n = 1$, $e^x \ge 1 + x$ by Exercise 75. Assume the statement is true for $n = k$. We need to prove the statement is true for $n = k + 1$. By the Induction Hypothesis,

$$e^x \ge 1 + x + x^2/2 + \cdots + x^k/k!.$$

Integrating both sides of this inequality yields

$$\int_0^x e^t\,dt = e^x - 1 \ge x + x^2/2 + \cdots + x^{k+1}/(k+1)!$$

or

$$e^x \ge 1 + x + x^2/2 + \cdots + x^{k+1}/(k+1)!$$

as required.

77. Use Exercise 75 to show that $e^x/x^2 \geq x/6$ and conclude that $\lim_{x\to\infty} e^x/x^2 = \infty$. Then use Exercise 76 to prove more generally that $\lim_{x\to\infty} e^x/x^n = \infty$ for all n.

SOLUTION By Exercise 75, $e^x \geq 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$. Thus

$$\frac{e^x}{x^2} \geq \frac{1}{x^2} + \frac{1}{x} + \frac{1}{2} + \frac{x}{6} \geq \frac{x}{6}.$$

Since $\lim_{x\to\infty} x/6 = \infty$, $\lim_{x\to\infty} e^x/x^2 = \infty$. More generally, by Exercise 76,

$$e^x \geq 1 + \frac{x^2}{2} + \cdots + \frac{x^{n+1}}{(n+1)!}.$$

Thus

$$\frac{e^x}{x^n} \geq \frac{1}{x^n} + \cdots + \frac{x}{(n+1)!} \geq \frac{x}{(n+1)!}.$$

Since $\lim_{x\to\infty} \frac{x}{(n+1)!} = \infty$, $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$.

Exercises 78–80 develop an elegant approach to the exponential and logarithm functions. Define a function $G(x)$ for $x > 0$:

$$G(x) = \int_1^x \frac{1}{t}\,dt$$

78. Defining ln *x* as an Integral This exercise proceeds as if we didn't know that $G(x) = \ln x$ and shows directly that $G(x)$ has all the basic properties of the logarithm. Prove the following statements.

(a) $\int_a^{ab} \frac{1}{t}\,dt = \int_1^b \frac{1}{t}\,dt$ for all $a, b > 0$. *Hint:* Use the substitution $u = t/a$.

(b) $G(ab) = G(a) + G(b)$. *Hint:* Break up the integral from 1 to ab into two integrals and use (a).

(c) $G(1) = 0$ and $G(a^{-1}) = -G(a)$ for $a > 0$.

(d) $G(a^n) = nG(a)$ for all $a > 0$ and integers n.

(e) $G(a^{1/n}) = \frac{1}{n}G(a)$ for all $a > 0$ and integers $n \neq 0$.

(f) $G(a^r) = rG(a)$ for all $a > 0$ and rational numbers r.

(g) $G(x)$ is increasing. *Hint:* Use FTC II.

(h) There exists a number a such that $G(a) > 1$. *Hint:* Show that $G(2) > 0$ and take $a = 2^m$ for $m > 1/G(2)$.

(i) $\lim_{x\to\infty} G(x) = \infty$ and $\lim_{x\to 0+} G(x) = -\infty$

(j) There exists a unique number E such that $G(E) = 1$.

(k) $G(E^r) = r$ for every rational number r.

SOLUTION

(a) Let $u = t/a$. Then $du = dt/a$, $u(a) = 1$, $u(ab) = b$ and

$$\int_a^{ab} \frac{1}{t}\,dt = \int_a^{ab} \frac{a}{at}\,dt = \int_1^b \frac{1}{u}\,du = \int_1^b \frac{1}{t}\,dt.$$

(b) Using part (a),

$$G(ab) = \int_1^{ab} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt = G(a) + G(b).$$

(c) First,

$$G(1) = \int_1^1 \frac{1}{t}\,dt = 0.$$

Next,

$$G(a^{-1}) = G\left(\frac{1}{a}\right) = \int_1^{1/a} \frac{1}{t}\,dt = \int_a^1 \frac{1}{t}\,dt \quad \text{by part (a) with } b = \frac{1}{a}$$

$$= -\int_1^a \frac{1}{t}\,dt = -G(a).$$

(d) Using part (a),

$$G(a^n) = \int_1^{a^n} \frac{1}{t}\,dt = \int_1^{a} \frac{1}{t}\,dt + \int_a^{a^2} \frac{1}{t}\,dt + \cdots + \int_{a^{n-1}}^{a^n} \frac{1}{t}\,dt$$
$$= \int_1^{a} \frac{1}{t}\,dt + \int_1^{a} \frac{1}{t}\,dt + \cdots + \int_1^{a} \frac{1}{t}\,dt = nG(a).$$

(e) $G(a) = G((a^{1/n})^n = nG(a^{1/n})$. Thus, $G(a^{1/n}) = \frac{1}{n}G(a)$.

(f) Let $r = m/n$ where m and n are integers. Then

$$\begin{aligned} G(a^r) &= G(a^{m/n}) = G((a^m)^{1/n}) \\ &= \frac{1}{n}G(a^m) \quad \text{by part (e)} \\ &= \frac{m}{n}G(a) \quad \text{by part d} \\ &= rG(a). \end{aligned}$$

(g) By the Fundamental Theorem of Calculus, $G(x)$ is continuous on $(0, \infty)$ and $G'(x) = \frac{1}{x} > 0$ for $x > 0$. Thus, $G(x)$ is increasing and one-to-one for $x > 0$.

(h) First note that

$$G(2) = \int_1^2 \frac{1}{t}\,dt > \frac{1}{2} > 0$$

because $\frac{1}{t} > \frac{1}{2}$ for $t \in (1, 2)$. Now, let $a = 2^m$ for m an integer greater than $1/G(2)$. Then

$$G(a) = G(2^m) = mG(2) > \frac{1}{G(2)} \cdot G(2) = 1.$$

(i) First, let a be the value from part (h) for which $G(a) > 1$ (note that a itself is greater than 1). Now,

$$\lim_{x\to\infty} G(x) = \lim_{m\to\infty} G(a^m) = G(a) \lim_{m\to\infty} m = \infty.$$

For the other limit, let $t = 1/x$ and note

$$\lim_{x\to 0+} G(x) = \lim_{t\to\infty} G\left(\frac{1}{t}\right) = -\lim_{t\to\infty} G(t) = -\infty.$$

(j) By part (c), $G(1) = 0$ and by part (h) there exists an a such that $G(a) > 1$. the Intermediate Value Theorem then guarantees there exists a number E such that $1 < E < a$ and $G(E) = 1$. We know that E is unique because G is one-to-one.

(k) Using part (f) and then part (j),

$$G(E^r) = rG(E) = r \cdot 1 = r.$$

79. Defining e^x Use Exercise 78 to prove the following statements.

(a) $G(x)$ has an inverse with domain **R** and range $\{x : x > 0\}$. Denote the inverse by $F(x)$.

(b) $F(x + y) = F(x)F(y)$ for all x, y. *Hint:* It suffices to show that $G(F(x)F(y)) = G(F(x + y))$.

(c) $F(r) = E^r$ for all numbers. In particular, $F(0) = 1$.

(d) $F'(x) = F(x)$. *Hint:* Use the formula for the derivative of an inverse function.

This shows that $E = e$ and $F(x)$ is the function e^x as defined in the text.

SOLUTION

(a) The domain of $G(x)$ is $x > 0$ and, by part (i) of the previous exercise, the range of $G(x)$ is **R**. Now,

$$G'(x) = \frac{1}{x} > 0$$

for all $x > 0$. Thus, $G(x)$ is increasing on its domain, which implies that $G(x)$ has an inverse. The domain of the inverse is **R** and the range is $\{x : x > 0\}$. Let $F(x)$ denote the inverse of $G(x)$.

(b) Let x and y be real numbers and suppose that $x = G(w)$ and $y = G(z)$ for some positive real numbers w and z. Then, using part (b) of the previous exercise

$$F(x + y) = F(G(w) + G(z)) = F(G(wz)) = wz = F(x) + F(y).$$

(c) Let r be any real number. By part (k) of the previous exercise, $G(E^r) = r$. By definition of an inverse function, it then follows that $F(r) = E^r$.

(d) By the formula for the derivative of an inverse function

$$F'(x) = \frac{1}{G'(F(x))} = \frac{1}{1/F(x)} = F(x).$$

80. Defining b^x Let $b > 0$ and let $f(x) = F(xG(b))$ with F as in Exercise 79. Use Exercise 78 (f) to prove that $f(r) = b^r$ for every rational number r. This gives us a way of defining b^x for irrational x, namely $b^x = f(x)$. With this definition, b^x is a differentiable function of x (because F is differentiable).

SOLUTION By Exercise 78 (f),

$$f(r) = F(rG(b)) = F(G(b^r)) = b^r,$$

for every rational number r.

81. The formula $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ is valid for $n \neq -1$. Show that the exceptional case $n = -1$ is a limit of the general case by applying L'Hôpital's Rule to the limit on the left.

$$\lim_{n\to -1} \int_1^x t^n\,dt = \int_1^x t^{-1}\,dt \qquad \text{(for fixed } x > 0)$$

Note that the integral on the left is equal to $\frac{x^{n+1} - 1}{n+1}$.

SOLUTION

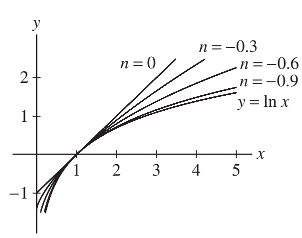
$$\lim_{n\to -1} \int_1^x t^n\,dt = \lim_{n\to -1} \left. \frac{t^{n+1}}{n+1} \right|_1^x = \lim_{n\to -1} \left(\frac{x^{n+1}}{n+1} - \frac{1^{n+1}}{n+1} \right)$$

$$= \lim_{n\to -1} \frac{x^{n+1} - 1}{n+1} = \lim_{n\to -1} (x^{n+1}) \ln x = \ln x = \int_1^x t^{-1}\,dt$$

Note that when using L'Hôpital's Rule in the second line, we need to differentiate with respect to n.

82. CAS The integral on the left in Exercise 81 is equal to $f_n(x) = \frac{x^{n+1} - 1}{n+1}$. Investigate the limit graphically by plotting $f_n(x)$ for $n = 0, -0.3, -0.6$, and -0.9 together with $\ln x$ on a single plot.

SOLUTION



83. **(a)** Explain why the shaded region in Figure 5 has area $\int_0^{\ln a} e^y\,dy$.

(b) Prove the formula $\int_1^a \ln x\,dx = a \ln a - \int_0^{\ln a} e^y\,dy$.

(c) Conclude that $\int_1^a \ln x\,dx = a \ln a - a + 1$.

(d) Use the result of (a) to find an antiderivative of $\ln x$.

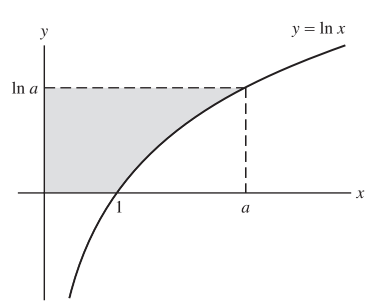


FIGURE 5

SOLUTION

(a) Interpreting the graph with y as the independent variable, we see that the function is $x = e^y$. Integrating in y then gives the area of the shaded region as $\int_0^{\ln a} e^y \, dy$

(b) We can obtain the area under the graph of $y = \ln x$ from $x = 1$ to $x = a$ by computing the area of the rectangle extending from $x = 0$ to $x = a$ horizontally and from $y = 0$ to $y = \ln a$ vertically and then subtracting the area of the shaded region. This yields

$$\int_1^a \ln x \, dx = a \ln a - \int_0^{\ln a} e^y \, dy.$$

(c) By direct calculation

$$\int_0^{\ln a} e^y \, dy = e^y \Big|_0^{\ln a} = a - 1.$$

Thus,

$$\int_1^a \ln x \, dx = a \ln a - (a - 1) = a \ln a - a + 1.$$

(d) Based on these results it appears that

$$\int \ln x \, dx = x \ln x - x + C.$$

5.8 Exponential Growth and Decay

Preliminary Questions

1. Two quantities increase exponentially with growth constants $k = 1.2$ and $k = 3.4$, respectively. Which quantity doubles more rapidly?

SOLUTION Doubling time is inversely proportional to the growth constant. Consequently, the quantity with $k = 3.4$ doubles more rapidly.

2. A cell population grows exponentially beginning with one cell. Which takes longer: increasing from one to two cells or increasing from 15 million to 20 million cells?

SOLUTION It takes longer for the population to increase from one cell to two cells, because this requires doubling the population. Increasing from 15 million to 20 million is less than doubling the population.

3. Referring to his popular book *A Brief History of Time*, the renowned physicist Stephen Hawking said, "Someone told me that each equation I included in the book would halve its sales." Find a differential equation satisfied by the function $S(n)$, the number of copies sold if the book has n equations.

SOLUTION Let $S(0)$ denote the sales with no equations in the book. Translating Hawking's observation into an equation yields

$$S(n) = \frac{S(0)}{2^n}.$$

Differentiating with respect to n then yields

$$\frac{dS}{dn} = S(0)\frac{d}{dn}2^{-n} = -\ln 2 S(0) 2^{-n} = -\ln 2 S(n).$$

4. The PV of N dollars received at time T is (choose the correct answer):

(a) The value at time T of N dollars invested today

(b) The amount you would have to invest today in order to receive N dollars at time T

SOLUTION The correct response is **(b)**: the PV of N dollars received at time T is the amount you would have to invest today in order to receive N dollars at time T.

5. In one year, you will be paid \$1. Will the PV increase or decrease if the interest rate goes up?

SOLUTION If the interest rate goes up, the present value of \$1 a year from now will decrease.

Exercises

1. A certain population P of bacteria obeys the exponential growth law $P(t) = 2000e^{1.3t}$ (t in hours).

(a) How many bacteria are present initially?

(b) At what time will there be 10,000 bacteria?

SOLUTION

(a) $P(0) = 2000e^0 = 2000$ bacteria initially.

(b) We solve $2000e^{1.3t} = 10,000$ for t. Thus, $e^{1.3t} = 5$ or

$$t = \frac{1}{1.3}\ln 5 \approx 1.24 \text{ hours.}$$

2. A quantity P obeys the exponential growth law $P(t) = e^{5t}$ (t in years).

(a) At what time t is $P = 10$?

(b) What is the doubling time for P?

SOLUTION

(a) $e^{5t} = 10$ when $t = \frac{1}{5}\ln 10 \approx 0.46$ years.

(b) The doubling time is $\frac{1}{5}\ln 2 \approx 0.14$ years.

3. Write $f(t) = 5(7)^t$ in the form $f(t) = P_0e^{kt}$ for some P_0 and k.

SOLUTION Because $7 = e^{\ln 7}$, it follows that

$$f(t) = 5(7)^t = 5(e^{\ln 7})^t = 5e^{t\ln 7}.$$

Thus, $P_0 = 5$ and $k = \ln 7$.

4. Write $f(t) = 9e^{1.4t}$ in the form $f(t) = P_0b^t$ for some P_0 and b.

SOLUTION Observe that

$$f(t) = 9e^{1.4t} = 9\left(e^{1.4}\right)^t,$$

so $P_0 = 9$ and $b = e^{1.4} \approx 4.0552$.

5. A certain RNA molecule replicates every 3 minutes. Find the differential equation for the number $N(t)$ of molecules present at time t (in minutes). How many molecules will be present after one hour if there is one molecule at $t = 0$?

SOLUTION The doubling time is $\frac{\ln 2}{k}$ so $k = \frac{\ln 2}{\text{doubling time}}$. Thus, the differential equation is $N'(t) = kN(t) = \frac{\ln 2}{3}N(t)$. With one molecule initially,

$$N(t) = e^{(\ln 2/3)t} = 2^{t/3}.$$

Thus, after one hour, there are

$$N(60) = 2^{60/3} = 1{,}048{,}576$$

molecules present.

6. A quantity P obeys the exponential growth law $P(t) = Ce^{kt}$ (t in years). Find the formula for $P(t)$, assuming that the doubling time is 7 years and $P(0) = 100$.

SOLUTION The doubling time is 7 years, so $7 = \ln 2/k$, or $k = \ln 2/7 = 0.099$ years^{-1}. With $P(0) = 100$, it follows that $P(t) = 100e^{0.099t}$.

7. Find all solutions to the differential equation $y' = -5y$. Which solution satisfies the initial condition $y(0) = 3.4$?

SOLUTION $y' = -5y$, so $y(t) = Ce^{-5t}$ for some constant C. The initial condition $y(0) = 3.4$ determines $C = 3.4$. Therefore, $y(t) = 3.4e^{-5t}$.

8. Find the solution to $y' = \sqrt{2}y$ satisfying $y(0) = 20$.

SOLUTION $y' = \sqrt{2}y$, so $y(t) = Ce^{\sqrt{2}t}$ for some constant C. The initial condition $y(0) = 20$ determines $C = 20$. Therefore, $y(t) = 20e^{\sqrt{2}t}$.

9. Find the solution to $y' = 3y$ satisfying $y(2) = 1000$.

SOLUTION $y' = 3y$, so $y(t) = Ce^{3t}$ for some constant C. The initial condition $y(2) = 1000$ determines $C = \frac{1000}{e^6}$. Therefore, $y(t) = \frac{1000}{e^6}e^{3t} = 1000e^{3(t-2)}$.

10. Find the function $y = f(t)$ that satisfies the differential equation $y' = -0.7y$ and the initial condition $y(0) = 10$.

SOLUTION Given that $y' = -0.7y$ and $y(0) = 10$, then $f(t) = 10e^{-0.7t}$.

11. The decay constant of cobalt-60 is 0.13 year^{-1}. Find its half-life.

SOLUTION Half-life $= \dfrac{\ln 2}{0.13} \approx 5.33$ years.

12. The half-life radium-226 is 1622 years. Find its decay constant.

SOLUTION Half-life $= \dfrac{\ln 2}{k}$ so $k = \dfrac{\ln 2}{\text{half-life}} = \dfrac{\ln 2}{1622} = 4.27 \times 10^{-4}\ \text{years}^{-1}$.

13. One of the world's smallest flowering plants, *Wolffia globosa* (Figure 13), has a doubling time of approximately 30 hours. Find the growth constant k and determine the initial population if the population grew to 1000 after 48 hours.

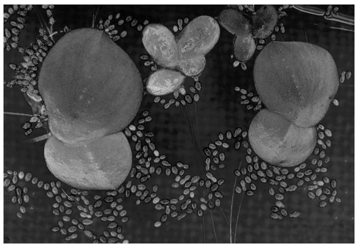


FIGURE 13 The tiny plants are *Wolffia*, with plant bodies smaller than the head of a pin.

SOLUTION By the formula for the doubling time, $30 = \dfrac{\ln 2}{k}$. Therefore,

$$k = \frac{\ln 2}{30} \approx 0.023\ \text{hours}^{-1}.$$

The plant population after t hours is $P(t) = P_0 e^{0.023t}$. If $P(48) = 1000$, then

$$P_0 e^{(0.023)48} = 1000 \quad \Rightarrow \quad P_0 = 1000e^{-(0.023)48} \approx 332$$

14. A 10-kg quantity of a radioactive isotope decays to 3 kg after 17 years. Find the decay constant of the isotope.

SOLUTION $P(t) = 10e^{-kt}$. Thus $P(17) = 3 = 10e^{-17k}$, so $k = \dfrac{\ln(3/10)}{-17} \approx 0.071\ \text{years}^{-1}$.

15. The population of a city is $P(t) = 2 \cdot e^{0.06t}$ (in millions), where t is measured in years. Calculate the time it takes for the population to double, to triple, and to increase seven-fold.

SOLUTION Since $k = 0.06$, the doubling time is

$$\frac{\ln 2}{k} \approx 11.55 \text{ years.}$$

The tripling time is calculated in the same way as the doubling time. Solve for Δ in the equation

$$\begin{aligned} P(t + \Delta) &= 3P(t) \\ 2 \cdot e^{0.06(t+\Delta)} &= 3(2e^{0.06t}) \\ 2 \cdot e^{0.06t} e^{0.06\Delta} &= 3(2e^{0.06t}) \\ e^{0.06\Delta} &= 3 \\ 0.06\Delta &= \ln 3, \end{aligned}$$

or $\Delta = \ln 3/0.06 \approx 18.31$ years. Working in a similar fashion, we find that the time required for the population to increase seven-fold is

$$\frac{\ln 7}{k} = \frac{\ln 7}{0.06} \approx 32.43 \text{ years.}$$

16. What is the differential equation satisfied by $P(t)$, the number of infected computer hosts in Example 4? Over which time interval would $P(t)$ increase one hundred-fold?

SOLUTION Because the rate constant is $k = 0.0815\ \text{s}^{-1}$, the differential equation for $P(t)$ is

$$\frac{dP}{dt} = 0.0815P.$$

The time for the number of infected computers to increase one hundred-fold is

$$\frac{\ln 100}{k} = \frac{\ln 100}{0.0815} \approx 56.51 \text{ s.}$$

17. The decay constant for a certain drug is $k = 0.35$ day^{-1}. Calculate the time it takes for the quantity present in the bloodstream to decrease by half, by one-third, and by one-tenth.

SOLUTION The time required for the quantity present in the bloodstream to decrease by half is

$$\frac{\ln 2}{k} = \frac{\ln 2}{0.35} \approx 1.98 \text{ days.}$$

To decay by one-third, the time is

$$\frac{\ln 3}{k} = \frac{\ln 3}{0.35} \approx 3.14 \text{ days.}$$

Finally, to decay by one-tenth, the time is

$$\frac{\ln 10}{k} = \frac{\ln 10}{0.35} \approx 6.58 \text{ days.}$$

18. Light Intensity The intensity of light passing through an absorbing medium decreases exponentially with the distance traveled. Suppose the decay constant for a certain plastic block is $k = 4$ m^{-1}. How thick must the block be to reduce the intensity by a factor of one-third?

SOLUTION Since intensity decreases exponentially, it can be modeled by an exponential decay equation $I(d) = I_0 e^{-kd}$. Assuming $I(0) = 1$, $I(d) = e^{-kd}$. Since the decay constant is $k = 4$, we have $I(d) = e^{-4d}$. Intensity will be reduced by a factor of one-third when $e^{-4d} = \frac{1}{3}$ or when $d = \dfrac{\ln(1/3)}{-4} \approx 0.275$ m.

19. Assuming that population growth is approximately exponential, which of the following two sets of data is most likely to represent the population (in millions) of a city over a 5-year period?

Year	2000	2001	2002	2003	2004
Set I	3.14	3.36	3.60	3.85	4.11
Set II	3.14	3.24	3.54	4.04	4.74

SOLUTION If the population growth is approximately exponential, then the ratio between successive years' data needs to be approximately the same.

Year	2000		2001		2002		2003		2004
Data I	3.14		3.36		3.60		3.85		4.11
Ratios		1.07006		1.07143		1.06944		1.06753	
Data II	3.14		3.24		3.54		4.04		4.74
Ratios		1.03185		1.09259		1.14124		1.17327	

As you can see, the ratio of successive years in the data from "Data I" is very close to 1.07. Therefore, we would expect exponential growth of about $P(t) \approx (3.14)(1.07^t)$.

20. The **atmospheric pressure** $P(h)$ (in kilopascals) at a height h meters above sea level satisfies a differential equation $P' = -kP$ for some positive constant k.

(a) Barometric measurements show that $P(0) = 101.3$ and $P(30, 900) = 1.013$. What is the decay constant k?

(b) Determine the atmospheric pressure at $h = 500$.

SOLUTION

(a) Because $P' = -kP$ for some positive constant k, $P(h) = Ce^{-kh}$ where $C = P(0) = 101.3$. Therefore, $P(h) = 101.3e^{-kh}$. We know that $P(30{,}900) = 101.3e^{-30{,}900k} = 1.013$. Solving for k yields

$$k = -\frac{1}{30{,}900} \ln\left(\frac{1.013}{101.3}\right) \approx 0.000149 \text{ meters}^{-1}.$$

(b) $P(500) = 101.3e^{-0.000149(500)} \approx 94.03$ kilopascals.

21. Degrees in Physics One study suggests that from 1955 to 1970, the number of bachelor's degrees in physics awarded per year by U.S. universities grew exponentially, with growth constant $k = 0.1$.
(a) If exponential growth continues, how long will it take for the number of degrees awarded per year to increase 14-fold?
(b) If 2500 degrees were awarded in 1955, in which year were 10,000 degrees awarded?

SOLUTION
(a) The time required for the number of degrees to increase 14-fold is

$$\frac{\ln 14}{k} = \frac{\ln 14}{0.1} \approx 26.39 \text{ years.}$$

(b) The doubling time is $(\ln 2)/0.1 \approx 0.693/0.1 = 6.93$ years. Since degrees are usually awarded once a year, we round off the doubling time to 7 years. The number quadruples after 14 years, so 10, 000 degrees would be awarded in 1969.

22. The **Beer–Lambert Law** is used in spectroscopy to determine the molar absorptivity α or the concentration c of a compound dissolved in a solution at low concentrations (Figure 14). The law states that the intensity I of light as it passes through the solution satisfies $\ln(I/I_0) = \alpha cx$, where I_0 is the initial intensity and x is the distance traveled by the light. Show that I satisfies a differential equation $dI/dx = -kI$ for some constant k.

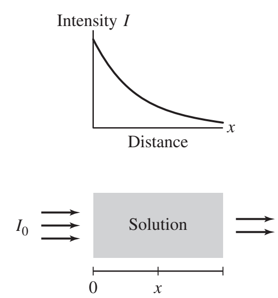


FIGURE 14 Light of intensity passing through a solution.

SOLUTION $\ln\left(\frac{I}{I_0}\right) = \alpha c I$ so $\frac{I}{I_0} = e^{\alpha c I}$ or $I = I_0 e^{\alpha c I}$. Therefore,

$$\frac{dI}{dx} = I_0 e^{\alpha c I}(\alpha c) = I(\alpha c) = -kI,$$

where $k = -\alpha c$ is a constant.

23. A sample of sheepskin parchment discovered by archaeologists had a C^{14}-to-C^{12} ratio equal to 40% of that found in the atmosphere. Approximately how old is the parchment?

SOLUTION The ratio of C^{14} to C^{12} is $Re^{-0.000121t} = 0.4R$ so $-0.000121t = \ln(0.4)$ or $t = 7572.65 \approx 7600$ years.

24. Chauvet Caves In 1994, three French speleologists (geologists specializing in caves) discovered a cave in southern France containing prehistoric cave paintings. A C^{14} analysis carried out by archeologist Helene Valladas showed the paintings to be between 29,700 and 32,400 years old, much older than any previously known human art. Given that the C^{14}-to-C^{12} ratio of the atmosphere is $R = 10^{-12}$, what range of C^{14}-to-C^{12} ratios did Valladas find in the charcoal specimens?

SOLUTION The C^{14}-C^{12} ratio found in the specimens ranged from

$$10^{-12}e^{-0.000121(32,400)} \approx 1.98 \times 10^{-14}$$

to

$$10^{-12}e^{-0.000121(29,700)} \approx 2.75 \times 10^{-14}.$$

25. A paleontologist discovers remains of animals that appear to have died at the onset of the Holocene ice age, between 10,000 and 12,000 years ago. What range of C^{14}-to-C^{12} ratio would the scientist expect to find in the animal remains?

SOLUTION The scientist would expect to find C^{14}-C^{12} ratios ranging from

$$10^{-12}e^{-0.000121(12,000)} \approx 2.34 \times 10^{-13}$$

to

$$10^{-12}e^{-0.000121(10,000)} \approx 2.98 \times 10^{-13}.$$

26. Inversion of Sugar When cane sugar is dissolved in water, it converts to invert sugar over a period of several hours. The percentage $f(t)$ of unconverted cane sugar at time t (in hours) satisfies $f' = -0.2f$. What percentage of cane sugar remains after 5 hours? After 10 hours?

SOLUTION $f' = -0.2f$, so $f(t) = Ce^{-0.2t}$. Since f is a percentage, at $t = 0$, $C = 100$ percent. Therefore. $f(t) = 100e^{-0.2t}$. Thus $f(5) = 100e^{-0.2(5)} \approx 36.79$ percent and $f(10) = 100e^{-0.2(10)} \approx 13.53$ percent.

27. Continuing with Exercise 26, suppose that 50 grams of sugar are dissolved in a container of water. After how many hours will 20 grams of invert sugar be present?

SOLUTION If there are 20 grams of invert sugar present, then there are 30 grams of unconverted sugar. This means that $f = 60$. Solving

$$100e^{-0.2t} = 60$$

for t yields

$$t = -\frac{1}{0.2}\ln 0.6 \approx 2.55 \text{ hours.}$$

28. Two bacteria colonies are cultivated in a laboratory. The first colony has a doubling time of 2 hours and the second a doubling time of 3 hours. Initially, the first colony contains 1000 bacteria and the second colony 3000 bacteria. At what time t will the sizes of the colonies be equal?

SOLUTION $P_1(t) = 1000e^{k_1 t}$ and $P_2(t) = 3000e^{k_2 t}$. Knowing that $k_1 = \frac{\ln 2}{2}$ hours^{-1} and $k_2 = \frac{\ln 2}{3}$ hours^{-1}, we need to solve $e^{k_1 t} = 3e^{k_2 t}$ for t. Thus

$$k_1 t = \ln(3e^{k_2 t}) = \ln 3 + \ln(e^{k_2 t}) = \ln 3 + k_2 t,$$

so

$$t = \frac{\ln 3}{k_1 - k_2} = \frac{6\ln 3}{\ln 2} \approx 9.51 \text{ hours.}$$

29. Moore's Law In 1965, Gordon Moore predicted that the number N of transistors on a microchip would increase exponentially.

(a) Does the table of data below confirm Moore's prediction for the period from 1971 to 2000? If so, estimate the growth constant k.

(b) CAS Plot the data in the table.

(c) Let $N(t)$ be the number of transistors t years after 1971. Find an approximate formula $N(t) \approx Ce^{kt}$, where t is the number of years after 1971.

(d) Estimate the doubling time in Moore's Law for the period from 1971 to 2000.

(e) How many transistors will a chip contain in 2015 if Moore's Law continues to hold?

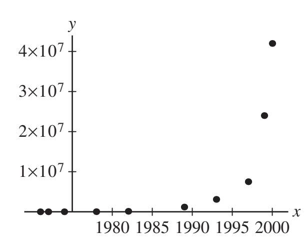
(f) Can Moore have expected his prediction to hold indefinitely?

Processor	Year	No. Transistors
4004	1971	2250
8008	1972	2500
8080	1974	5000
8086	1978	29,000
286	1982	120,000
386 processor	1985	275,000
486 DX processor	1989	1,180,000
Pentium processor	1993	3,100,000
Pentium II processor	1997	7,500,000
Pentium III processor	1999	24,000,000
Pentium 4 processor	2000	42,000,000
Xeon processor	2008	1,900,000,000

SOLUTION

(a) Yes, the graph looks like an exponential graph especially towards the latter years. We estimate the growth constant by setting 1971 as our starting point, so $P_0 = 2250$. Therefore, $P(t) = 2250e^{kt}$. In 2008, $t = 37$. Therefore, $P(37) = 2250e^{37k} = 1{,}900{,}000{,}000$, so $k = \frac{\ln 844{,}444.444}{37} \approx 0.369$. Note: A better estimate can be found by calculating k for each time period and then averaging the k values.

(b)



(c) $N(t) = 2250e^{0.369t}$
(d) The doubling time is $\ln 2/0.369 \approx 1.88$ years.
(e) In 2015, $t = 44$ years. Therefore, $N(44) = 2250e^{0.369(44)} \approx 2.53 \times 10^{10}$.
(f) No, you can't make a microchip smaller than an atom.

30. Assume that in a certain country, the rate at which jobs are created is proportional to the number of people who already have jobs. If there are 15 million jobs at $t = 0$ and 15.1 million jobs 3 months later, how many jobs will there be after 2 years?

SOLUTION Let $J(t)$ denote the number of people, in millions, who have jobs at time t, in months. Because the rate at which jobs are created is proportional to the number of people who already have jobs, $J'(t) = kJ(t)$, for some constant k. Given that $J(0) = 15$, it then follows that $J(t) = 15e^{kt}$. To determine k, we use $J(3) = 15.1$; therefore,

$$k = \frac{1}{3}\ln\left(\frac{15.1}{15}\right) \approx 2.215 \times 10^{-3} \text{ months}^{-1}.$$

Finally, after two years, there are

$$J(24) = 15e^{0.002215(24)} \approx 15.8 \text{ million}$$

jobs.

31. The only functions with a *constant* doubling time are the exponential functions P_0e^{kt} with $k > 0$. Show that the doubling time of linear function $f(t) = at + b$ at time t_0 is $t_0 + b/a$ (which increases with t_0). Compute the doubling times of $f(t) = 3t + 12$ at $t_0 = 10$ and $t_0 = 20$.

SOLUTION Let $f(t) = at + b$ and suppose $f(t_0) = P_0$. The time it takes for the value of f to double is the solution of the equation

$$2P_0 = 2(at_0 + b) = at + b \quad \text{or} \quad t = 2t_0 + b/a.$$

For the function $f(t) = 3t + 12$, $a = 3$, $b = 12$ and $b/a = 4$. With $t_0 = 10$, the doubling time is then 24; with $t_0 = 20$, the doubling time is 44.

32. Verify that the half-life of a quantity that decays exponentially with decay constant k is equal to $(\ln 2)/k$.

SOLUTION Let $y = Ce^{-kt}$ be an exponential decay function. Let t be the half-life of the quantity y, that is, the time t when $y = \frac{C}{2}$. Solving $\frac{C}{2} = Ce^{-kt}$ for t we get $-\ln 2 = -kt$, so $t = \ln 2/k$.

33. Compute the balance after 10 years if \$2000 is deposited in an account paying 9% interest and interest is compounded (a) quarterly, (b) monthly, and (c) continuously.

SOLUTION

(a) $P(10) = 2000(1 + 0.09/4)^{4(10)} = \$4870.38$
(b) $P(10) = 2000(1 + 0.09/12)^{12(10)} = \$4902.71$
(c) $P(10) = 2000e^{0.09(10)} = \$4919.21$

34. Suppose \$500 is deposited into an account paying interest at a rate of 7%, continuously compounded. Find a formula for the value of the account at time t. What is the value of the account after 3 years?

SOLUTION Let $P(t)$ denote the value of the account at time t. Because the initial deposit is \$500 and the account pays interest at a rate of 7%, compounded continuously, it follows that $P(t) = 500e^{0.07t}$. After three years, the value of the account is $P(3) = 500e^{0.07(3)} = \$616.84$.

35. A bank pays interest at a rate of 5%. What is the yearly multiplier if interest is compounded

(a) three times a year? **(b)** continuously?

SOLUTION

(a) $P(t) = P_0\left(1 + \frac{0.05}{3}\right)^{3t}$, so the yearly multiplier is $\left(1 + \frac{0.05}{3}\right)^{3} \approx 1.0508$.
(b) $P(t) = P_0e^{0.05t}$, so the yearly multiplier is $e^{0.05} \approx 1.0513$.

36. How long will it take for \$4000 to double in value if it is deposited in an account bearing 7% interest, continuously compounded?

SOLUTION The doubling time is $\frac{\ln 2}{0.7} \approx 9.9$ years.

37. How much must one invest today in order to receive \$20,000 after 5 years if interest is compounded continuously at the rate $r = 9\%$?

SOLUTION Solving $20{,}000 = P_0e^{0.09(5)}$ for P_0 yields

$$P_0 = \frac{20{,}000}{e^{0.45}} \approx \$12{,}752.56.$$

38. An investment increases in value at a continuously compounded rate of 9%. How large must the initial investment be in order to build up a value of \$50,000 over a 7-year period?

SOLUTION Solving $50{,}000 = P_0 e^{0.09(7)}$ for P_0 yields

$$P_0 = \frac{50{,}000}{e^{0.63}} \approx \$26{,}629.59.$$

39. Compute the PV of \$5000 received in 3 years if the interest rate is (a) 6% and (b) 11%. What is the PV in these two cases if the sum is instead received in 5 years?

SOLUTION In 3 years:

(a) $PV = 5000e^{-0.06(3)} = \$4176.35$

(b) $PV = 5000e^{-0.11(3)} = \$3594.62$

In 5 years:

(a) $PV = 5000e^{-0.06(5)} = \$3704.09$

(b) $PV = 5000e^{-0.11(5)} = \$2884.75$

40. Is it better to receive \$1000 today or \$1300 in 4 years? Consider $r = 0.08$ and $r = 0.03$.

SOLUTION Assuming continuous compounding, if $r = 0.08$, then the present value of \$1300 four years from now is $1300e^{-0.08(4)} = \$943.99$. It is better to get \$1000 now. On the other hand, if $r = 0.03$, the present value of \$1300 four years from now is $1300e^{-0.03(4)} = \$1153.00$, so it is better to get the \$1,300 in four years.

41. Find the interest rate r if the PV of \$8000 to be received in 1 year is \$7300.

SOLUTION Solving $7300 = 8000e^{-r(1)}$ for r yields

$$r = -\ln\left(\frac{7300}{8000}\right) = 0.0916,$$

or 9.16%.

42. A company can earn additional profits of \$500,000/year for 5 years by investing \$2 million to upgrade its factory. Is the investment worthwhile if the interest rate is 6%? (Assume the savings are received as a lump sum at the end of each year.)

SOLUTION The present value of the stream of additional profits is

$$500{,}000(e^{-0.06} + e^{-0.12} + e^{-0.18} + e^{-0.24} + e^{-0.3}) = \$2{,}095{,}700.63.$$

This is more than the \$2 million cost of the upgrade, so the upgrade should be made.

43. A new computer system costing \$25,000 will reduce labor costs by \$7000/year for 5 years.

(a) Is it a good investment if $r = 8\%$?

(b) How much money will the company actually save?

SOLUTION

(a) The present value of the reduced labor costs is

$$7000(e^{-0.08} + e^{-0.16} + e^{-0.24} + e^{-0.32} + e^{-0.4}) = \$27{,}708.50.$$

This is more than the \$25,000 cost of the computer system, so the computer system should be purchased.

(b) The present value of the savings is

$$\$27{,}708.50 - \$25{,}000 = \$2708.50.$$

44. After winning \$25 million in the state lottery, Jessica learns that she will receive five yearly payments of \$5 million beginning immediately.

(a) What is the PV of Jessica's prize if $r = 6\%$?

(b) How much more would the prize be worth if the entire amount were paid today?

SOLUTION

(a) The present value of the prize is

$$5{,}000{,}000(e^{-0.24} + e^{-0.18} + e^{-0.12} + e^{-0.06} + e^{-0.06(0)}) = \$22{,}252{,}915.21.$$

(b) If the entire amount were paid today, the present value would be \$25 million, or \$2,747,084.79 more than the stream of payments made over five years.

45. Use Eq. (3) to compute the PV of an income stream paying out $R(t) = \$5000$/year continuously for 10 years, assuming $r = 0.05$.

SOLUTION $PV = \int_0^{10} 5000e^{-0.05t}\,dt = -100{,}000e^{-0.05t}\Big|_0^{10} = \$39{,}346.93.$

46. Find the PV of an investment that pays out continuously at a rate of \$800/year for 5 years, assuming $r = 0.08$.

SOLUTION $PV = \int_0^{5} 800e^{-0.08t}\,dt = -10{,}000e^{-0.08t}\Big|_0^{5} = \$3296.80.$

47. Find the PV of an income stream that pays out continuously at a rate $R(t) = \$5000e^{0.1t}$/year for 7 years, assuming $r = 0.05$.

SOLUTION $PV = \int_0^{7} 5000e^{0.1t}e^{-0.05t}\,dt = \int_0^{7} 5000e^{0.05t}\,dt = 100{,}000e^{0.05t}\Big|_0^{7} = \$41{,}906.75.$

48. A commercial property generates income at the rate $R(t)$. Suppose that $R(0) = \$70{,}000$/year and that $R(t)$ increases at a continuously compounded rate of 5%. Find the PV of the income generated in the first 4 years if $r = 6\%$.

SOLUTION $PV = \int_0^{4} 70{,}000e^{0.05t}e^{-0.06t}\,dt = -\frac{70{,}000}{0.01}e^{-0.01t}\Big|_0^{4} = \$274{,}473.93.$

49. Show that an investment that pays out R dollars per year continuously for T years has a PV of $R(1 - e^{-rT})/r$.

SOLUTION The present value of an investment that pays out R dollars/year continuously for T years is

$$PV = \int_0^{T} Re^{-rt}\,dt.$$

Let $u = -rt,\ du = -r\,dt$. Then

$$PV = -\frac{1}{r}\int_0^{-rT} Re^{u}\,du = -\frac{R}{r}e^{u}\Big|_0^{-rT} = -\frac{R}{r}(e^{-rT} - 1) = \frac{R}{r}(1 - e^{-rT}).$$

50. Explain this statement: If T is very large, then the PV of the income stream described in Exercise 49 is approximately R/r.

SOLUTION Because

$$\lim_{T\to\infty} e^{-rT} = \lim_{T\to\infty} \frac{1}{e^{rt}} = 0,$$

it follows that

$$\lim_{T\to\infty} \frac{R}{r}(1 - e^{-rT}) = \frac{R}{r}.$$

51. Suppose that $r = 0.06$. Use the result of Exercise 50 to estimate the payout rate R needed to produce an income stream whose PV is \$20,000, assuming that the stream continues for a large number of years.

SOLUTION From Exercise 50, $PV = \frac{R}{r}$ so $20{,}000 = \frac{R}{0.06}$ or $R = \$1200$.

52. Verify by differentiation:

$$\int te^{-rt}\,dt = -\frac{e^{-rt}(1 + rt)}{r^2} + C \qquad \boxed{5}$$

Use Eq. (5) to compute the PV of an investment that pays out income continuously at a rate $R(t) = (5000 + 1000t)$ dollars per year for 5 years, assuming $r = 0.05$.

SOLUTION

$$\frac{d}{dt}\left(-\frac{e^{-rt}(1 + rt)}{r^2}\right) = \frac{-1}{r^2}\left(e^{-rt}(r) + (1 + rt)(-re^{-rt})\right) = \frac{-1}{r}\left(e^{-rt} - e^{-rt} - rte^{-rt}\right) = te^{-rt}$$

Therefore

$$PV = \int_0^{5} (5000 + 1000t)e^{-0.05t}\,dt = \int_0^{5} 5000e^{-0.05t}\,dt + \int_0^{5} 1000te^{-0.05t}\,dt$$

$$= \frac{5000}{-0.05}(e^{-0.05(5)} - 1) - 1000\left(\frac{e^{-0.05(5)}(1 + 0.05(5))}{(0.05)^2}\right) + 1000\frac{1}{(0.05)^2}$$

$$= 22{,}119.92 - 389{,}400.39 + 400{,}000 \approx \$32{,}719.53.$$

53. Use Eq. (5) to compute the PV of an investment that pays out income continuously at a rate $R(t) = (5000 + 1000t)e^{0.02t}$ dollars per year for 10 years, assuming $r = 0.08$.

SOLUTION

$$PV = \int_0^{10} (5000 + 1000t)(e^{0.02t})e^{-0.08t}\,dt = \int_0^{10} 5000e^{-0.06t}\,dt + \int_0^{10} 1000te^{-0.06t}\,dt$$
$$= \frac{5000}{-0.06}(e^{-0.06(10)} - 1) - 1000\left(\frac{e^{-0.06(10)}(1 + 0.06(10))}{(0.06)^2}\right) + 1000\frac{1}{(0.06)^2}$$
$$= 37{,}599.03 - 243{,}916.28 + 277{,}777.78 \approx \$71{,}460.53.$$

54. **Banker's Rule of 70** If you earn an interest rate of R percent, continuously compounded, your money doubles after approximately $70/R$ years. For example, at $R = 5\%$, your money doubles after $70/5$ or 14 years. Use the concept of doubling time to justify the Banker's Rule. (*Note:* Sometimes, the rule $72/R$ is used. It is less accurate but easier to apply because 72 is divisible by more numbers than 70.)

SOLUTION The doubling time is

$$t = \frac{\ln 2}{r} = \frac{\ln 2 \cdot 100}{r\%} = \frac{69.93}{r\%} \approx \frac{70}{r\%}.$$

55. **Drug Dosing Interval** Let $y(t)$ be the drug concentration (in mg/kg) in a patient's body at time t. The initial concentration is $y(0) = L$. Additional doses that increase the concentration by an amount d are administered at regular time intervals of length T. In between doses, $y(t)$ decays exponentially—that is, $y' = -ky$. Find the value of T (in terms of k and d) for which the the concentration varies between L and $L - d$ as in Figure 15.

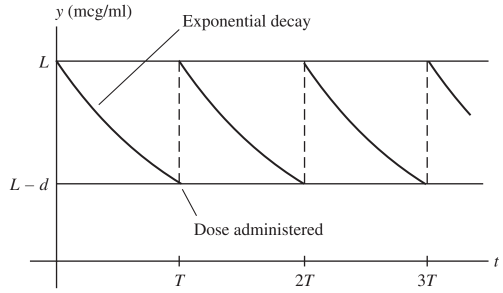


FIGURE 15 Drug concentration with periodic doses.

SOLUTION Because $y' = -ky$ and $y(0) = L$, it follows that $y(t) = Le^{-kt}$. We want $y(T) = L - d$, thus

$$Le^{-kT} = L - d \quad \text{or} \quad T = -\frac{1}{k}\ln\left(1 - \frac{d}{L}\right).$$

*Exercises 56 and 57: The **Gompertz differential equation***

$$\frac{dy}{dt} = ky\ln\left(\frac{y}{M}\right) \tag{6}$$

(where M and k are constants) was introduced in 1825 by the English mathematician Benjamin Gompertz and is still used today to model aging and mortality.

56. Show that $y = Me^{ae^{kt}}$ satisfies Eq. (6) for any constant a.

SOLUTION Let $y = Me^{ae^{kt}}$. Then

$$\frac{dy}{dt} = M(kae^{kt})e^{ae^{kt}}$$

and, since

$$\ln(y/M) = ae^{kt},$$

we have

$$ky\ln(y/M) = Mkae^{kt}e^{ae^{kt}} = \frac{dy}{dt}.$$

57. To model mortality in a population of 200 laboratory rats, a scientist assumes that the number $P(t)$ of rats alive at time t (in months) satisfies Eq. (6) with $M = 204$ and $k = 0.15$ month^{-1} (Figure 16). Find $P(t)$ [note that $P(0) = 200$] and determine the population after 20 months.

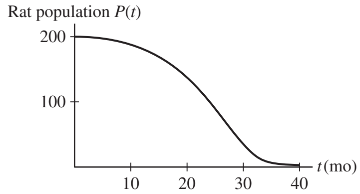


FIGURE 16

SOLUTION The solution to the Gompertz equation with $M = 204$ and $k = 0.15$ is of the form:

$$P(t) = 204e^{ae^{0.15t}}$$

Applying the initial condition allows us to solve for a:

$$200 = 204e^{a}$$
$$\frac{200}{204} = e^{a}$$
$$\ln\left(\frac{200}{204}\right) = a$$

so that $a \approx -0.02$. After $t = 20$ months,

$$P(20) = 204e^{-0.02e^{0.15(20)}} = 136.51,$$

so there are 136 rats.

58. **Isotopes for Dating** Which of the following would be most suitable for dating extremely old rocks: carbon-14 (half-life 5570 years), lead-210 (half-life 22.26 years), or potassium-49 (half-life 1.3 billion years)? Explain why.

SOLUTION For extremely old rocks, you need to have an isotope that decays very slowly. In other words, you want a very large half-life such as Potassium-49; otherwise, the amount of undecayed isotope in the rock sample would be too small to accurately measure.

59. Let $P = P(t)$ be a quantity that obeys an exponential growth law with growth constant k. Show that P increases m-fold after an interval of $(\ln m)/k$ years.

SOLUTION For m-fold growth, $P(t) = mP_0$ for some t. Solving $mP_0 = P_0e^{kt}$ for t, we find $t = \frac{\ln m}{k}$

Further Insights and Challenges

60. **Average Time of Decay** Physicists use the radioactive decay law $R = R_0e^{-kt}$ to compute the average or *mean time* M until an atom decays. Let $F(t) = R/R_0 = e^{-kt}$ be the fraction of atoms that have survived to time t without decaying.

(a) Find the inverse function $t(F)$.

(b) By definition of $t(F)$, a fraction $1/N$ of atoms decays in the time interval

$$\left[t\left(\frac{j}{N}\right), t\left(\frac{j-1}{N}\right)\right]$$

Use this to justify the approximation $M \approx \frac{1}{N}\sum_{j=1}^{N} t\left(\frac{j}{N}\right)$. Then argue, by passing to the limit as $N \to \infty$, that $M = \int_0^1 t(F)\,dF$. Strictly speaking, this is an *improper integral* because $t(0)$ is infinite (it takes an infinite amount of time for all atoms to decay). Therefore, we define M as a limit

$$M = \lim_{c\to 0}\int_c^1 t(F)\,dF$$

(c) Verify the formula $\int \ln x\,dx = x\ln x - x$ by differentiation and use it to show that for $c > 0$,

$$M = \lim_{c\to 0}\left(\frac{1}{k} + \frac{1}{k}(c\ln c - c)\right)$$

(d) Show that $M = 1/k$ by evaluating the limit (use L'Hôpital's Rule to compute $\lim_{c\to 0} c \ln c$).

(e) What is the mean time to decay for radon (with a half-life of 3.825 days)?

SOLUTION

(a) $F = e^{-kt}$ so $\ln F = -kt$ and $t(F) = \dfrac{\ln F}{-k}$

(b) $M \approx \dfrac{1}{N}\sum_{j=1}^{N} t(j/N)$. For the interval [0, 1], from the approximation given, the subinterval length is $1/N$ and thus the right-hand endpoints have x-coordinate (j/N). Thus we have a Riemann sum and by definition,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} t(j/N) = \int_0^1 t(F)\,dF.$$

(c) $\dfrac{d}{dx}(x\ln x - x) = x\left(\dfrac{1}{x}\right) + \ln x - 1 = \ln x$. Thus

$$\int_c^1 t(F)\,dF = -\frac{1}{k}(F\ln F - F)\Big|_c^1 = \frac{1}{k}(F - F\ln F)\Big|_c^1$$
$$= \frac{1}{k}(1 - 1\ln 1 - (c - c\ln c))$$
$$= \frac{1}{k} + \frac{1}{k}(c\ln c - c).$$

(d) By, L'Hôpital's Rule,

$$\lim_{c\to 0+} c\ln c = \lim_{c\to 0+} \frac{\ln c}{c^{-1}} = \lim_{c\to 0+} \frac{c^{-1}}{-c^{-2}} = -\lim_{c\to 0+} c = 0.$$

Thus, $M = \lim_{c\to 0}\int_c^1 t(F)\,dF = \lim_{c\to 0}\left(\dfrac{1}{k} + \dfrac{1}{k}(c\ln c - c)\right) = \dfrac{1}{k}$.

(e) Since the half-life is 3.825 days, $k = \dfrac{\ln 2}{3.825}$ and $\dfrac{1}{k} = 5.52$. Thus, $M = 5.52$ days.

61. Modify the proof of the relation $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ given in the text to prove $e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$. *Hint:* Express $\ln(1 + xn^{-1})$ as an integral and estimate above and below by rectangles.

SOLUTION Start by expressing

$$\ln\left(1 + \frac{x}{n}\right) = \int_1^{1+x/n} \frac{dt}{t}.$$

Following the proof in the text, we note that

$$\frac{x}{n+x} \le \ln\left(1 + \frac{x}{n}\right) \le \frac{x}{n}$$

provided $x > 0$, while

$$\frac{x}{n} \le \ln\left(1 + \frac{x}{n}\right) \le \frac{x}{n+x}$$

when $x < 0$. Multiplying both sets of inequalities by n and passing to the limit as $n \to \infty$, the squeeze theorem guarantees that

$$\lim_{n\to\infty}\left(\ln\left(1 + \frac{x}{n}\right)\right)^n = x.$$

Finally,

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$

62. Prove that, for $n > 0$,

$$\left(1 + \frac{1}{n}\right)^n \le e \le \left(1 + \frac{1}{n}\right)^{n+1}$$

Hint: Take logarithms and use Eq. (4).

SOLUTION Taking logarithms throughout the desired inequality, we find the equivalent inequality

$$n\ln\left(1 + \frac{1}{n}\right) \le 1 \le (n+1)\ln\left(1 + \frac{1}{n}\right).$$

Multiplying Eq. (4) by n yields

$$\frac{n}{n+1} \le n \ln\left(1+\frac{1}{n}\right) \le 1,$$

which establishes the left-hand side of the desired inequality. On the other hand, multiplying Eq. (4) by $n+1$ yields

$$1 \le (n+1)\ln\left(1+\frac{1}{n}\right) \le 1+\frac{1}{n},$$

which establishes the right-hand side of the desired inequality.

63. A bank pays interest at the rate r, compounded M times yearly. The **effective interest rate** r_e is the rate at which interest, if compounded annually, would have to be paid to produce the same yearly return.

(a) Find r_e if $r = 9\%$ compounded monthly.
(b) Show that $r_e = (1+r/M)^M - 1$ and that $r_e = e^r - 1$ if interest is compounded continuously.
(c) Find r_e if $r = 11\%$ compounded continuously.
(d) Find the rate r that, compounded weekly, would yield an effective rate of 20%.

SOLUTION

(a) Compounded monthly, $P(t) = P_0(1+r/12)^{12t}$. By the definition of r_e,

$$P_0(1+0.09/12)^{12t} = P_0(1+r_e)^t$$

so

$$(1+0.09/12)^{12t} = (1+r_e)^t \quad \text{or} \quad r_e = (1+0.09/12)^{12} - 1 = 0.0938,$$

or 9.38%

(b) In general,

$$P_0(1+r/M)^{Mt} = P_0(1+r_e)^t,$$

so $(1+r/M)^{Mt} = (1+r_e)^t$ or $r_e = (1+r/M)^M - 1$. If interest is compounded continuously, then $P_0e^{rt} = P_0(1+r_e)^t$ so $e^{rt} = (1+r_e)^t$ or $r_e = e^r - 1$.

(c) Using part (b), $r_e = e^{0.11} - 1 \approx 0.1163$ or 11.63%.

(d) Solving

$$0.20 = \left(1+\frac{r}{52}\right)^{52} - 1$$

for r yields $r = 52(1.2^{1/52} - 1) = 0.1826$ or 18.26%.

CHAPTER REVIEW EXERCISES

In Exercises 1–4, refer to the function $f(x)$ whose graph is shown in Figure 1.

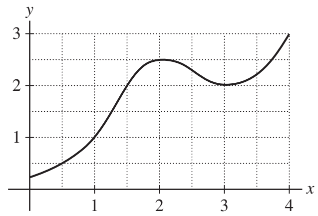


FIGURE 1

1. Estimate L_4 and M_4 on [0, 4].

SOLUTION With $n = 4$ and an interval of [0, 4], $\Delta x = \frac{4-0}{4} = 1$. Then,

$$L_4 = \Delta x(f(0) + f(1) + f(2) + f(3)) = 1\left(\frac{1}{4} + 1 + \frac{5}{2} + 2\right) = \frac{23}{4}$$

and

$$M_4 = \Delta x\left(f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{7}{2}\right)\right) = 1\left(\frac{1}{2} + 2 + \frac{9}{4} + \frac{9}{4}\right) = 7.$$

2. Estimate R_4, L_4, and M_4 on [1, 3].

SOLUTION With $n = 4$ and an interval of [1, 3], $\Delta x = \frac{3-1}{4} = \frac{1}{2}$. Then,

$$R_4 = \Delta x\left(f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) + f(3)\right) = \frac{1}{2}\left(2 + \frac{5}{2} + \frac{9}{4} + 2\right) = \frac{35}{8};$$

$$L_4 = \Delta x\left(f(1) + f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right)\right) = \frac{1}{2}\left(1 + 2 + \frac{5}{2} + \frac{9}{4}\right) = \frac{31}{8};\ \text{and}$$

$$M_4 = \Delta x\left(f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right)\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{9}{4} + \frac{5}{2} + \frac{17}{8}\right) = \frac{67}{16}.$$

3. Find an interval $[a, b]$ on which R_4 is larger than $\int_a^b f(x)\,dx$. Do the same for L_4.

SOLUTION In general, R_N is larger than $\int_a^b f(x)\,dx$ on any interval $[a, b]$ over which $f(x)$ is increasing. Given the graph of $f(x)$, we may take $[a, b] = [0, 2]$. In order for L_4 to be larger than $\int_a^b f(x)\,dx$, $f(x)$ must be decreasing over the interval $[a, b]$. We may therefore take $[a, b] = [2, 3]$.

4. Justify $\frac{3}{2} \le \int_1^2 f(x)\,dx \le \frac{9}{4}$.

SOLUTION Because $f(x)$ is increasing on [1, 2], we know that

$$L_N \le \int_1^2 f(x)\,dx \le R_N$$

for any N. Now,

$$L_2 = \frac{1}{2}(1 + 2) = \frac{3}{2} \qquad \text{and} \qquad R_2 = \frac{1}{2}\left(2 + \frac{5}{2}\right) = \frac{9}{4},$$

so

$$\frac{3}{2} \le \int_1^2 f(x)\,dx \le \frac{9}{4}.$$

In Exercises 5–8, let $f(x) = x^2 + 3x$.

5. Calculate R_6, M_6, and L_6 for $f(x)$ on the interval [2, 5]. Sketch the graph of $f(x)$ and the corresponding rectangles for each approximation.

SOLUTION Let $f(x) = x^2 + 3x$. A uniform partition of [2, 5] with $N = 6$ subintervals has

$$\Delta x = \frac{5-2}{6} = \frac{1}{2}, \qquad x_j = a + j\Delta x = 2 + \frac{j}{2},$$

and

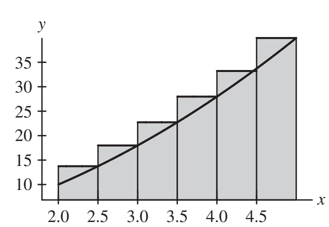
$$x_j^* = a + \left(j - \frac{1}{2}\right)\Delta x = \frac{7}{4} + \frac{j}{2}.$$

Now,

$$R_6 = \Delta x \sum_{j=1}^{6} f(x_j) = \frac{1}{2}\left(f\left(\frac{5}{2}\right) + f(3) + f\left(\frac{7}{2}\right) + f(4) + f\left(\frac{9}{2}\right) + f(5)\right)$$

$$= \frac{1}{2}\left(\frac{55}{4} + 18 + \frac{91}{4} + 28 + \frac{135}{4} + 40\right) = \frac{625}{8}.$$

The rectangles corresponding to this approximation are shown below.

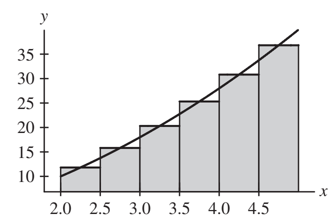


Next,

$$M_6 = \Delta x \sum_{j=1}^{6} f(x_j^*) = \frac{1}{2}\left(f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right) + f\left(\frac{17}{4}\right) + f\left(\frac{19}{4}\right)\right)$$

$$= \frac{1}{2}\left(\frac{189}{16} + \frac{253}{16} + \frac{325}{16} + \frac{405}{16} + \frac{493}{16} + \frac{589}{16}\right) = \frac{2254}{32} = \frac{1127}{16}.$$

The rectangles corresponding to this approximation are shown below.

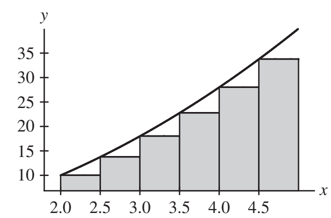


Finally,

$$L_6 = \Delta x \sum_{j=0}^{5} f(x_j) = \frac{1}{2}\left(f(2) + f\left(\frac{5}{2}\right) + f(3) + f\left(\frac{7}{2}\right) + f(4) + f\left(\frac{9}{2}\right)\right)$$

$$= \frac{1}{2}\left(10 + \frac{55}{4} + 18 + \frac{91}{4} + 28 + \frac{135}{4}\right) = \frac{505}{8}.$$

The rectangles corresponding to this approximation are shown below.



6. Use FTC I to evaluate $A(x) = \int_{-2}^{x} f(t)\,dt$.

SOLUTION Let $f(x) = x^2 + 3x$. Then

$$A(x) = \int_{-2}^{x} (t^2 + 3t)\,dt = \left(\frac{1}{3}t^3 + \frac{3}{2}t^2\right)\Big|_{-2}^{x} = \frac{1}{3}x^3 + \frac{3}{2}x^2 - \left(-\frac{8}{3} + 6\right) = \frac{1}{3}x^3 + \frac{3}{2}x^2 - \frac{10}{3}.$$

7. Find a formula for R_N for $f(x)$ on [2, 5] and compute $\int_2^5 f(x)\,dx$ by taking the limit.

SOLUTION Let $f(x) = x^2 + 3x$ on the interval [2, 5]. Then $\Delta x = \frac{5-2}{N} = \frac{3}{N}$ and $a = 2$. Hence,

$$R_N = \Delta x \sum_{j=1}^{N} f(2 + j\Delta x) = \frac{3}{N}\sum_{j=1}^{N}\left(\left(2 + \frac{3j}{N}\right)^2 + 3\left(2 + \frac{3j}{N}\right)\right) = \frac{3}{N}\sum_{j=1}^{N}\left(10 + \frac{21j}{N} + \frac{9j^2}{N^2}\right)$$

$$= 30 + \frac{63}{N^2}\sum_{j=1}^{N} j + \frac{27}{N^3}\sum_{j=1}^{N} j^2$$

$$= 30 + \frac{63}{N^2}\left(\frac{N^2}{2} + \frac{N}{2}\right) + \frac{27}{N^3}\left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right)$$

$$= \frac{141}{2} + \frac{45}{N} + \frac{9}{2N^2}$$

and

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty}\left(\frac{141}{2} + \frac{45}{N} + \frac{9}{2N^2}\right) = \frac{141}{2}.$$

8. Find a formula for L_N for $f(x)$ on [0, 2] and compute $\int_0^2 f(x)\,dx$ by taking the limit.

SOLUTION Let $f(x) = x^2 + 3x$ and N be a positive integer. Then

$$\Delta x = \frac{2-0}{N} = \frac{2}{N}$$

and

$$x_j = a + j\Delta x = 0 + \frac{2j}{N} = \frac{2j}{N}$$

for $0 \le j \le N$. Thus,

$$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j) = \frac{2}{N}\sum_{j=0}^{N-1}\left(\frac{4j^2}{N^2} + \frac{6j}{N}\right) = \frac{8}{N^3}\sum_{j=0}^{N-1} j^2 + \frac{12}{N^2}\sum_{j=0}^{N-1} j$$
$$= \frac{4(N-1)(2N-1)}{3N^2} + \frac{6(N-1)}{N} = \frac{26}{3} - \frac{10}{N} + \frac{4}{3N^2}.$$

Finally,

$$\int_0^2 f(x)\,dx = \lim_{N\to\infty}\left(\frac{26}{3} - \frac{10}{N} + \frac{4}{3N^2}\right) = \frac{26}{3}.$$

9. Calculate R_5, M_5, and L_5 for $f(x) = (x^2+1)^{-1}$ on the interval [0, 1].

SOLUTION Let $f(x) = (x^2+1)^{-1}$. A uniform partition of [0, 1] with $N = 5$ subintervals has

$$\Delta x = \frac{1-0}{5} = \frac{1}{5}, \qquad x_j = a + j\Delta x = \frac{j}{5},$$

and

$$x_j^* = a + \left(j - \frac{1}{2}\right)\Delta x = \frac{2j-1}{10}.$$

Now,

$$R_5 = \Delta x \sum_{j=1}^{5} f(x_j) = \frac{1}{5}\left(f\left(\frac{1}{5}\right) + f\left(\frac{2}{5}\right) + f\left(\frac{3}{5}\right) + f\left(\frac{4}{5}\right) + f(1)\right)$$
$$= \frac{1}{5}\left(\frac{25}{26} + \frac{25}{29} + \frac{25}{34} + \frac{25}{41} + \frac{1}{2}\right) \approx 0.733732.$$

Next,

$$M_5 = \Delta x \sum_{j=1}^{5} f(x_j^*) = \frac{1}{5}\left(f\left(\frac{1}{10}\right) + f\left(\frac{3}{10}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{7}{10}\right) + f\left(\frac{9}{10}\right)\right)$$
$$= \frac{1}{5}\left(\frac{100}{101} + \frac{100}{109} + \frac{4}{5} + \frac{100}{149} + \frac{100}{181}\right) \approx 0.786231.$$

Finally,

$$L_5 = \Delta x \sum_{j=0}^{4} f(x_j) = \frac{1}{5}\left(f(0) + f\left(\frac{1}{5}\right) + f\left(\frac{2}{5}\right) + f\left(\frac{3}{5}\right) + f\left(\frac{4}{5}\right)\right)$$
$$= \frac{1}{5}\left(1 + \frac{25}{26} + \frac{25}{29} + \frac{25}{34} + \frac{25}{41}\right) \approx 0.833732.$$

10. Let R_N be the Nth right-endpoint approximation for $f(x) = x^3$ on [0, 4] (Figure 2).

(a) Prove that $R_N = \dfrac{64(N+1)^2}{N^2}$.

(b) Prove that the area of the region within the right-endpoint rectangles above the graph is equal to

$$\frac{64(2N+1)}{N^2}$$

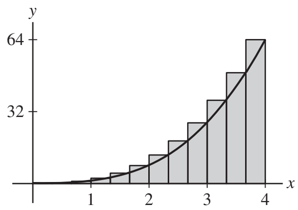


FIGURE 2 Approximation R_N for $f(x) = x^3$ on [0, 4].

SOLUTION

(a) Let $f(x) = x^3$ and N be a positive integer. Then

$$\Delta x = \frac{4-0}{N} = \frac{4}{N} \qquad \text{and} \qquad x_j = a + j\Delta x = 0 + \frac{4j}{N} = \frac{4j}{N}$$

for $0 \le j \le N$. Thus,

$$R_N = \Delta x \sum_{j=1}^{N} f(x_j) = \frac{4}{N} \sum_{j=1}^{N} \frac{64j^3}{N^3} = \frac{256}{N^4} \sum_{j=1}^{N} j^3 = \frac{256}{N^4} \frac{N^2(N+1)^2}{4} = \frac{64(N+1)^2}{N^2}.$$

(b) The area between the graph of $y = x^3$ and the x-axis over [0, 4] is

$$\int_0^4 x^3\,dx = \frac{1}{4}x^4\Big|_0^4 = 64.$$

The area of the region below the right-endpoint rectangles and above the graph is therefore

$$\frac{64(N+1)^2}{N^2} - 64 = \frac{64(2N+1)}{N^2}.$$

11. Which approximation to the area is represented by the shaded rectangles in Figure 3? Compute R_5 and L_5.

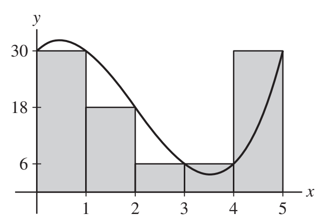


FIGURE 3

SOLUTION There are five rectangles and the height of each is given by the function value at the right endpoint of the subinterval. Thus, the area represented by the shaded rectangles is R_5.

From the figure, we see that $\Delta x = 1$. Then

$$R_5 = 1(30 + 18 + 6 + 6 + 30) = 90 \qquad \text{and} \qquad L_5 = 1(30 + 30 + 18 + 6 + 6) = 90.$$

12. Calculate any two Riemann sums for $f(x) = x^2$ on the interval [2, 5], but choose partitions with at least five subintervals of unequal widths and intermediate points that are neither endpoints nor midpoints.

SOLUTION Let $f(x) = x^2$. Riemann sums will, of course, vary. Here are two possibilities. Take $N = 5$,

$$P = \{x_0 = 2, x_1 = 2.7, x_2 = 3.1, x_3 = 3.6, x_4 = 4.2, x_5 = 5\}$$

and

$$C = \{c_1 = 2.5, c_2 = 3, c_3 = 3.5, c_4 = 4, c_5 = 4.5\}.$$

Then,

$$R(f, P, C) = \sum_{j=1}^{5} \Delta x_j f(c_j) = 0.7(6.25) + 0.4(9) + 0.5(12.25) + 0.6(16) + 0.8(20.25) = 39.9.$$

Alternately, take $N = 6$,

$$P = \{x_0 = 2, x_1 = 2.5, x_2 = 3.5, x_3 = 4, x_4 = 4.25, x_5 = 4.75, x_6 = 5\}$$

and

$$C = \{c_1 = 2.1, c_2 = 3, c_3 = 3.7, c_4 = 4.2, c_5 = 4.5, c_6 = 4.8\}.$$

Then,

$$\begin{aligned} R(f, P, C) &= \sum_{j=1}^{6} \Delta x_j f(c_j) \\ &= 0.5(4.41) + 1(9) + 0.5(13.69) + 0.25(17.64) + 0.5(20.25) + 0.25(23.04) = 38.345. \end{aligned}$$

In Exercises 13–16, express the limit as an integral (or multiple of an integral) and evaluate.

13. $\displaystyle\lim_{N\to\infty} \frac{\pi}{6N} \sum_{j=1}^{N} \sin\left(\frac{\pi}{3} + \frac{\pi j}{6N}\right)$

SOLUTION Let $f(x) = \sin x$ and N be a positive integer. A uniform partition of the interval $[\pi/3, \pi/2]$ with N subintervals has

$$\Delta x = \frac{\pi}{6N} \qquad \text{and} \qquad x_j = \frac{\pi}{3} + \frac{\pi j}{6N}$$

for $0 \le j \le N$. Then

$$\frac{\pi}{6N} \sum_{j=1}^{N} \sin\left(\frac{\pi}{3} + \frac{\pi j}{6N}\right) = \Delta x \sum_{j=1}^{N} f(x_j) = R_N;$$

consequently,

$$\lim_{N\to\infty} \frac{\pi}{6N} \sum_{j=1}^{N} \sin\left(\frac{\pi}{3} + \frac{\pi j}{6N}\right) = \int_{\pi/3}^{\pi/2} \sin x\, dx = -\cos x\Big|_{\pi/3}^{\pi/2} = 0 + \frac{1}{2} = \frac{1}{2}.$$

14. $\displaystyle\lim_{N\to\infty} \frac{3}{N} \sum_{k=0}^{N-1} \left(10 + \frac{3k}{N}\right)$

SOLUTION Let $f(x) = x$ and N be a positive integer. A uniform partition of the interval [10, 13] with N subintervals has

$$\Delta x = \frac{3}{N} \qquad \text{and} \qquad x_j = 10 + \frac{3j}{N}$$

for $0 \le j \le N$. Then

$$\frac{3}{N} \sum_{k=0}^{N-1} \left(10 + \frac{3k}{N}\right) = \Delta x \sum_{j=0}^{N-1} f(x_j) = L_N;$$

consequently,

$$\begin{aligned} \lim_{N\to\infty} \frac{3}{N} \sum_{k=0}^{N-1} \left(10 + \frac{3k}{N}\right) &= \int_{10}^{13} x\, dx = \frac{1}{2}x^2\Big|_{10}^{13} \\ &= \frac{169}{2} - \frac{100}{2} = \frac{69}{2}. \end{aligned}$$

15. $\lim_{N\to\infty} \frac{5}{N} \sum_{j=1}^{N} \sqrt{4+5j/N}$

SOLUTION Let $f(x) = \sqrt{x}$ and N be a positive integer. A uniform partition of the interval [4, 9] with N subintervals has

$$\Delta x = \frac{5}{N} \qquad \text{and} \qquad x_j = 4 + \frac{5j}{N}$$

for $0 \le j \le N$. Then

$$\frac{5}{N} \sum_{j=1}^{N} \sqrt{4+5j/N} = \Delta x \sum_{j=1}^{N} f(x_j) = R_N;$$

consequently,

$$\lim_{N\to\infty} \frac{5}{N} \sum_{j=1}^{N} \sqrt{4+5j/N} = \int_4^9 \sqrt{x}\, dx = \frac{2}{3} x^{3/2} \Big|_4^9 = \frac{54}{3} - \frac{16}{3} = \frac{38}{3}.$$

16. $\lim_{N\to\infty} \frac{1^k + 2^k + \cdots + N^k}{N^{k+1}} \quad (k > 0)$

SOLUTION Observe that

$$\frac{1^k + 2^k + 3^k + \cdots + N^k}{N^{k+1}} = \frac{1}{N}\left[\left(\frac{1}{N}\right)^k + \left(\frac{2}{N}\right)^k + \left(\frac{3}{N}\right)^k + \cdots \left(\frac{N}{N}\right)^k\right] = \frac{1}{N} \sum_{j=1}^{N} \left(\frac{j}{N}\right)^k.$$

Now, let $f(x) = x^k$ and N be a positive integer. A uniform partition of the interval [0, 1] with N subintervals has

$$\Delta x = \frac{1}{N} \qquad \text{and} \qquad x_j = \frac{j}{N}$$

for $0 \le j \le N$. Then

$$\frac{1}{N} \sum_{j=1}^{N} \left(\frac{j}{N}\right)^k = \Delta x \sum_{j=1}^{N} f(x_j) = R_N;$$

consequently,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \left(\frac{j}{N}\right)^k = \int_0^1 x^k\, dx = \frac{1}{k+1} x^{k+1} \Big|_0^1 = \frac{1}{k+1}.$$

In Exercises 17–20, use the given substitution to evaluate the integral.

17. $\int_0^2 \frac{dt}{4t+12}, \quad u = 4t + 12$

SOLUTION Let $u = 4t + 12$. Then $du = 4dt$, and the new limits of integration are $u = 12$ and $u = 20$. Thus,

$$\int_0^2 \frac{dt}{4t+12} = \frac{1}{4} \int_{12}^{20} \frac{du}{u} = \frac{1}{4} \ln u \Big|_{12}^{20} = \frac{1}{4}(\ln 20 - \ln 12) = \frac{1}{4} \ln \frac{20}{12} = \frac{1}{4} \ln \frac{5}{3}.$$

18. $\int \frac{(x^2+1)\, dx}{(x^3+3x)^4}, \qquad u = x^3 + 3x$

SOLUTION Let $u = x^3 + 3x$. Then $du = (3x^2 + 3)\, dx = 3(x^2 + 1)\, dx$ and

$$\int \frac{(x^2+1)\, dx}{(x^3+3x)^4} = \frac{1}{3} \int u^{-4}\, du = -\frac{1}{9} u^{-3} + C = -\frac{1}{9}(x^3 + 3x)^{-3} + C.$$

19. $\displaystyle\int_0^{\pi/6} \sin x \cos^4 x\,dx, \qquad u = \cos x$

SOLUTION Let $u = \cos x$. Then $du = -\sin x\,dx$ and the new limits of integration are $u = 1$ and $u = \sqrt{3}/2$. Thus,

$$\int_0^{\pi/6} \sin x \cos^4 x\,dx = -\int_1^{\sqrt{3}/2} u^4\,du = -\frac{1}{5}u^5\Big|_1^{\sqrt{3}/2} = \frac{1}{5}\left(1 - \frac{9\sqrt{3}}{32}\right).$$

20. $\displaystyle\int \sec^2(2\theta)\tan(2\theta)\,d\theta, \quad u = \tan(2\theta)$

SOLUTION Let $u = \tan(2\theta)$. Then $du = 2\sec^2(2\theta)\,d\theta$ and

$$\int \sec^2(2\theta)\tan(2\theta)\,d\theta = \frac{1}{2}\int u\,du = \frac{1}{4}u^2 + C = \frac{1}{4}\tan^2(2\theta) + C.$$

In Exercises 21–70, evaluate the integral.

21. $\displaystyle\int (20x^4 - 9x^3 - 2x)\,dx$

SOLUTION $\displaystyle\int (20x^4 - 9x^3 - 2x)\,dx = 4x^5 - \frac{9}{4}x^4 - x^2 + C.$

22. $\displaystyle\int_0^2 (12x^3 - 3x^2)\,dx$

SOLUTION $\displaystyle\int_0^2 (12x^3 - 3x^2)\,dx = (3x^4 - x^3)\Big|_0^2 = (48 - 8) - 0 = 40.$

23. $\displaystyle\int (2x^2 - 3x)^2\,dx$

SOLUTION $\displaystyle\int (2x^2 - 3x)^2\,dx = \int (4x^4 - 12x^3 + 9x^2)\,dx = \frac{4}{5}x^5 - 3x^4 + 3x^3 + C.$

24. $\displaystyle\int_0^1 (x^{7/3} - 2x^{1/4})\,dx$

SOLUTION $\displaystyle\int_0^1 (x^{7/3} - 2x^{1/4})\,dx = \left(\frac{3}{10}x^{10/3} - \frac{8}{5}x^{5/4}\right)\Big|_0^1 = \frac{3}{10} - \frac{8}{5} = -\frac{13}{10}.$

25. $\displaystyle\int \frac{x^5 + 3x^4}{x^2}\,dx$

SOLUTION $\displaystyle\int \frac{x^5 + 3x^4}{x^2}\,dx = \int (x^3 + 3x^2)\,dx = \frac{1}{4}x^4 + x^3 + C.$

26. $\displaystyle\int_1^3 r^{-4}\,dr$

SOLUTION $\displaystyle\int_1^3 r^{-4}\,dr = -\frac{1}{3}r^{-3}\Big|_1^3 = -\frac{1}{3}\left(\frac{1}{27} - 1\right) = \frac{26}{81}.$

27. $\int_{-3}^{3} |x^2 - 4|\, dx$

SOLUTION

$$\begin{aligned}\int_{-3}^{3} |x^2 - 4|\, dx &= \int_{-3}^{2} (x^2 - 4)\, dx + \int_{-2}^{2} (4 - x^2)\, dx + \int_{2}^{3} (x^2 - 4)\, dx \\ &= \left(\frac{1}{3}x^3 - 4x\right)\Big|_{-3}^{-2} + \left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^{2} + \left(\frac{1}{3}x^3 - 4x\right)\Big|_{2}^{3} \\ &= \left(\frac{16}{3} - 3\right) + \left(\frac{16}{3} + \frac{16}{3}\right) + \left(-3 + \frac{16}{3}\right) \\ &= \frac{46}{3}.\end{aligned}$$

28. $\int_{-2}^{4} |(x-1)(x-3)|\, dx$

SOLUTION

$$\begin{aligned}\int_{-2}^{4} |(x-1)(x-3)|\, dx &= \int_{-2}^{1} (x^2 - 4x + 3)\, dx + \int_{1}^{3} (-x^2 + 4x - 3)\, dx + \int_{3}^{4} (x^2 - 4x + 3)\, dx \\ &= \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_{-2}^{1} + \left(-\frac{1}{3}x^3 + 2x^2 - 3x\right)\Big|_{1}^{3} + \left(\frac{1}{3}x^3 - 2x^2 + 3x\right)\Big|_{3}^{4} \\ &= \frac{4}{3} - \left(-\frac{50}{3}\right) + 0 - \left(-\frac{4}{3}\right) + \frac{4}{3} - 0 \\ &= \frac{62}{3}.\end{aligned}$$

29. $\int_{1}^{3} [t]\, dt$

SOLUTION

$$\int_{1}^{3} [t]\, dt = \int_{1}^{2} [t]\, dt + \int_{2}^{3} [t]\, dt = \int_{1}^{2} dt + \int_{2}^{3} 2\, dt = t\Big|_{1}^{2} + 2t\Big|_{2}^{3} = (2 - 1) + (6 - 4) = 3.$$

30. $\int_{0}^{2} (t - [t])^2\, dt$

SOLUTION

$$\begin{aligned}\int_{0}^{2} (t - [t])^2\, dt &= \int_{0}^{1} t^2\, dt + \int_{1}^{2} (t-1)^2\, dt \\ &= \frac{1}{3}t^3\Big|_{0}^{1} + \frac{1}{3}(t-1)^3\Big|_{1}^{2} \\ &= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.\end{aligned}$$

31. $\int (10t - 7)^{14}\, dt$

SOLUTION Let $u = 10t - 7$. Then $du = 10dt$ and

$$\int (10t - 7)^{14}\, dt = \frac{1}{10}\int u^{14}\, du = \frac{1}{150}u^{15} + C = \frac{1}{150}(10t - 7)^{15} + C.$$

32. $\int_{2}^{3} \sqrt{7y - 5}\, dy$

SOLUTION Let $u = 7y - 5$. Then $du = 7dy$ and when $y = 2$, $u = 9$ and when $y = 3$, $u = 16$. Finally,

$$\int_{2}^{3} \sqrt{7y - 5}\, dy = \frac{1}{7}\int_{9}^{16} u^{1/2}\, du = \frac{1}{7} \cdot \frac{2}{3}u^{3/2}\Big|_{9}^{16} = \frac{2}{21}(64 - 27) = \frac{74}{21}.$$

33. $\displaystyle\int \frac{(2x^3+3x)\,dx}{(3x^4+9x^2)^5}$

SOLUTION Let $u = 3x^4 + 9x^2$. Then $du = (12x^3 + 18x)\,dx = 6(2x^3 + 3x)\,dx$ and

$$\int \frac{(2x^3+3x)\,dx}{(3x^4+9x^2)^5} = \frac{1}{6}\int u^{-5}\,du = -\frac{1}{24}u^{-4} + C = -\frac{1}{24}(3x^4+9x^2)^{-4} + C.$$

34. $\displaystyle\int_{-3}^{-1} \frac{x\,dx}{(x^2+5)^2}$

SOLUTION Let $u = x^2 + 5$. Then $du = 2x\,dx$ and

$$\int_{-3}^{-1} \frac{x\,dx}{(x^2+5)^2} = \frac{1}{2}\int_{14}^{6} u^{-2}\,du = -\frac{1}{2}u^{-1}\Big|_{14}^{6}$$
$$= -\frac{1}{2}\left(\frac{1}{6} - \frac{1}{14}\right) = -\frac{1}{21}.$$

35. $\displaystyle\int_0^5 15x\sqrt{x+4}\,dx$

SOLUTION Let $u = x + 4$. Then $x = u - 4$, $du = dx$ and the new limits of integration are $u = 4$ and $u = 9$. Thus,

$$\int_0^5 15x\sqrt{x+4}\,dx = \int_4^9 15(u-4)\sqrt{u}\,du$$
$$= 15\int_4^9 (u^{3/2} - 4u^{1/2})\,du$$
$$= 15\left(\frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2}\right)\Big|_4^9$$
$$= 15\left(\left(\frac{486}{5} - 72\right) - \left(\frac{64}{5} - \frac{64}{3}\right)\right)$$
$$= 506.$$

36. $\displaystyle\int t^2\sqrt{t+8}\,dt$

SOLUTION Let $u = t + 8$. Then $du = dt$, $t = u - 8$, and

$$\int t^2\sqrt{t+8}\,dt = \int (u-8)^2\sqrt{u}\,du = \int (u^{5/2} - 16u^{3/2} + 64u^{1/2})\,du$$
$$= \frac{2}{7}u^{7/2} - \frac{32}{5}u^{5/2} + \frac{128}{3}u^{3/2} + C$$
$$= \frac{2}{7}(t+8)^{7/2} - \frac{32}{5}(t+8)^{5/2} + \frac{128}{3}(t+8)^{3/2} + C.$$

37. $\displaystyle\int_0^1 \cos\left(\frac{\pi}{3}(t+2)\right)\,dt$

SOLUTION $\displaystyle\int_0^1 \cos\left(\frac{\pi}{3}(t+2)\right)\,dt = \frac{3}{\pi}\sin\left(\frac{\pi}{3}(t+2)\right)\Big|_0^1 = -\frac{3\sqrt{3}}{2\pi}.$

38. $\displaystyle\int_{\pi/2}^{\pi} \sin\left(\frac{5\theta-\pi}{6}\right)\,d\theta$

SOLUTION Let

$$u = \frac{5\theta - \pi}{6} \quad \text{so that} \quad du = \frac{5}{6}d\theta.$$

Then

$$\int_{\pi/2}^{\pi} \sin\left(\frac{5\theta-\pi}{6}\right)\,d\theta = \frac{6}{5}\int_{\pi/4}^{2\pi/3} \sin u\,du$$
$$= -\frac{6}{5}\cos u\Big|_{\pi/4}^{2\pi} 3$$
$$= -\frac{6}{5}\left(-\frac{1}{2} - \frac{\sqrt{2}}{2}\right) = \frac{3}{5}(1+\sqrt{2}).$$

39. $\int t^2 \sec^2(9t^3+1)\,dt$

SOLUTION Let $u = 9t^3 + 1$. Then $du = 27t^2\,dt$ and

$$\int t^2 \sec^2(9t^3+1)\,dt = \frac{1}{27}\int \sec^2 u\,du = \frac{1}{27}\tan u + C = \frac{1}{27}\tan(9t^3+1) + C.$$

40. $\int \sin^2(3\theta)\cos(3\theta)\,d\theta$

SOLUTION Let $u = \sin(3\theta)$. Then $du = 3\cos(3\theta)d\theta$ and

$$\int \sin^2(3\theta)\cos(3\theta)\,d\theta = \frac{1}{3}\int u^2\,du = \frac{1}{9}u^3 + C = \frac{1}{9}\sin^3(3\theta) + C.$$

41. $\int \csc^2(9-2\theta)\,d\theta$

SOLUTION Let $u = 9 - 2\theta$. Then $du = -2\,d\theta$ and

$$\int \csc^2(9-2\theta)\,d\theta = -\frac{1}{2}\int \csc^2 u\,du = \frac{1}{2}\cot u + C = \frac{1}{2}\cot(9-2\theta) + C.$$

42. $\int \sin\theta\sqrt{4-\cos\theta}\,d\theta$

SOLUTION Let $u = 4 - \cos\theta$. Then $du = \sin\theta\,d\theta$ and

$$\int \sin\theta\sqrt{4-\cos\theta}\,d\theta = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(4-\cos\theta)^{3/2} + C.$$

43. $\int_0^{\pi/3} \frac{\sin\theta}{\cos^{2/3}\theta}\,d\theta$

SOLUTION Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$ and when $\theta = 0$, $u = 1$ and when $\theta = \frac{\pi}{3}$, $u = \frac{1}{2}$. Finally,

$$\int_0^{\pi/3} \frac{\sin\theta}{\cos^{2/3}\theta}\,d\theta = -\int_1^{1/2} u^{-2/3}\,du = -3u^{1/3}\Big|_1^{1/2} = -3(2^{-1/3}-1) = 3 - \frac{3\sqrt[3]{4}}{2}.$$

44. $\int \frac{\sec^2 t\,dt}{(\tan t - 1)^2}$

SOLUTION Let $u = \tan t - 1$. Then $du = \sec^2 t\,dt$ and

$$\int \frac{\sec^2 t\,dt}{(\tan t - 1)^2} = \int u^{-2}\,du = -u^{-1} + C = -\frac{1}{\tan t - 1} + C.$$

45. $\int e^{9-2x}\,dx$

SOLUTION Let $u = 9 - 2x$. Then $du = -2\,dx$, and

$$\int e^{9-2x}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{9-2x} + C.$$

46. $\int_1^3 e^{4x-3}\,dx$

SOLUTION $\int_1^3 e^{4x-3}\,dx = \frac{1}{4}e^{4x-3}\Big|_1^3 = \frac{1}{4}(e^9 - e).$

47. $\int x^2 e^{x^3}\,dx$

SOLUTION Let $u = x^3$. Then $du = 3x^2\,dx$, and

$$\int x^2 e^{x^3}dx = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C.$$

48. $\displaystyle\int_0^{\ln 3} e^{x-e^x}\,dx$

SOLUTION Note $e^{x-e^x} = e^x e^{-e^x}$. Now, let $u = e^x$. Then $du = e^x\,dx$, and the new limits of integration are $u = e^0 = 1$ and $u = e^{\ln 3} = 3$. Thus,

$$\int_0^{\ln 3} e^{x-e^x}\,dx = \int_0^{\ln 3} e^x e^{-e^x}\,dx = \int_1^3 e^{-u}\,du = -e^{-t}\Big|_1^3 = -(e^{-3} - e^{-1}) = e^{-1} - e^{-3}.$$

49. $\displaystyle\int e^x 10^x\,dx$

SOLUTION $\displaystyle\int e^x 10^x\,dx = \int (10e)^x\,dx = \frac{(10e)^x}{\ln(10e)} + C = \frac{(10e)^x}{\ln 10 + \ln e} + C = \frac{10^x e^x}{\ln 10 + 1} + C.$

50. $\displaystyle\int e^{-2x}\sin(e^{-2x})\,dx$

SOLUTION Let $u = e^{-2x}$. Then $du = -2e^{-2x}dx$, and

$$\int e^{-2x}\sin\left(e^{-2x}\right)\,dx = -\frac{1}{2}\int \sin u\,du = \frac{\cos u}{2} + C = \frac{1}{2}\cos\left(e^{-2x}\right) + C.$$

51. $\displaystyle\int \frac{e^{-x}\,dx}{(e^{-x}+2)^3}$

SOLUTION Let $u = e^{-x} + 2$. Then $du = -e^{-x}\,dx$ and

$$\int \frac{e^{-x}\,dx}{(e^{-x}+2)^3} = -\int u^{-3}\,du = \frac{1}{2u^2} + C = \frac{1}{2(e^{-x}+2)^2} + C.$$

52. $\displaystyle\int \sin\theta\cos\theta e^{\cos^2\theta+1}\,d\theta$

SOLUTION Let $u = \cos^2\theta + 1$. Then $du = -2\sin\theta\cos\theta\,d\theta$ and

$$\int \sin\theta\cos\theta e^{\cos^2\theta+1}\,d\theta = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{\cos^2\theta+1} + C.$$

53. $\displaystyle\int_0^{\pi/6} \tan 2\theta\,d\theta$

SOLUTION $\displaystyle\int_0^{\pi/6} \tan 2\theta\,d\theta = \frac{1}{2}\ln|\sec 2\theta|\Big|_0^{\pi/6} = \frac{1}{2}\ln 2.$

54. $\displaystyle\int_{\pi/3}^{2\pi/3} \cot\left(\frac{1}{2}\theta\right)\,d\theta$

SOLUTION

$$\begin{aligned}\int_{\pi/3}^{2\pi/3} \cot\left(\frac{1}{2}\theta\right)\,d\theta &= 2\ln\left|\sin\frac{\theta}{2}\right|\Big|_{\pi/3}^{2\pi}3\\ &= 2\left(\ln\sin\frac{\pi}{3} - \ln\sin\frac{\pi}{6}\right)\\ &= 2\left(\ln\frac{\sqrt{3}}{2} - \ln\frac{1}{2}\right) = \ln 3.\end{aligned}$$

55. $\displaystyle\int \frac{dt}{t(1+(\ln t)^2)}$

SOLUTION Let $u = \ln t$. Then, $du = \frac{1}{t}\,dt$ and

$$\int \frac{dt}{t(1+(\ln t)^2)} = \int \frac{du}{1+u^2} = \tan^{-1}u + C = \tan^{-1}(\ln t) + C.$$

56. $\displaystyle\int \frac{\cos(\ln x)\,dx}{x}$

SOLUTION Let $u = \ln x$. Then $du = \frac{dx}{x}$, and

$$\int \frac{\cos(\ln x)}{x}dx = \int \cos u\,du = \sin u + C = \sin(\ln x) + C.$$

57. $\displaystyle\int_1^e \frac{\ln x\,dx}{x}$

SOLUTION Let $u = \ln x$. Then $du = \frac{dx}{x}$ and the new limits of integration are $u = \ln 1 = 0$ and $u = \ln e = 1$. Thus,

$$\int_1^e \frac{\ln x\,dx}{x} = \int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}.$$

58. $\displaystyle\int \frac{dx}{x\sqrt{\ln x}}$

SOLUTION Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$, and

$$\int \frac{dx}{x\sqrt{\ln x}} = \int u^{-1/2}\,du = 2\sqrt{u} + C = 2\sqrt{\ln x} + C.$$

59. $\displaystyle\int \frac{dx}{4x^2+9}$

SOLUTION Let $u = \frac{2x}{3}$. Then $x = \frac{3}{2}u$, $dx = \frac{3}{2}\,du$, and

$$\int \frac{dx}{4x^2+9} = \int \frac{\frac{3}{2}\,du}{4\cdot\frac{9}{4}u^2+9} = \frac{1}{6}\int \frac{du}{u^2+1} = \frac{1}{6}\tan^{-1}u + C = \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right) + C.$$

60. $\displaystyle\int_0^{0.8} \frac{dx}{\sqrt{1-x^2}}$

SOLUTION $\displaystyle\int_0^{0.8} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big|_0^{0.8} = \sin^{-1} 0.8 - \sin^{-1} 0 = \sin^{-1} 0.8.$

61. $\displaystyle\int_4^{12} \frac{dx}{x\sqrt{x^2-1}}$

SOLUTION $\displaystyle\int_4^{12} \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x\Big|_4^{12} = \sec^{-1}12 - \sec^{-1}4.$

62. $\displaystyle\int_0^3 \frac{x\,dx}{x^2+9}$

SOLUTION Let $u = x^2+9$. Then $du = 2x\,dx$, and the new limits of integration are $u = 9$ and $u = 18$. Thus,

$$\int_0^3 \frac{x\,dx}{x^2+9} = \frac{1}{2}\int_9^{18} \frac{du}{u} = \frac{1}{2}\ln u\Big|_9^{18} = \frac{1}{2}(\ln 18 - \ln 9) = \frac{1}{2}\ln\frac{18}{9} = \frac{1}{2}\ln 2.$$

63. $\displaystyle\int_0^3 \frac{dx}{x^2+9}$

SOLUTION Let $u = \frac{x}{3}$. Then $du = \frac{dx}{3}$, and the new limits of integration are $u = 0$ and $u = 1$. Thus,

$$\int_0^3 \frac{dx}{x^2+9} = \frac{1}{3}\int_0^1 \frac{dt}{t^2+1} = \frac{1}{3}\tan^{-1}t\Big|_0^1 = \frac{1}{3}(\tan^{-1}1 - \tan^{-1}0) = \frac{1}{3}\left(\frac{\pi}{4}-0\right) = \frac{\pi}{12}.$$

64. $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$

SOLUTION Let $u = e^x$. Then

$$du = e^x\,dx \quad\Rightarrow\quad du = u\,dx \quad\Rightarrow\quad u^{-1}du = dx$$

By substitution, we obtain

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{du}{u\sqrt{u^2-1}}$$
$$= \sec^{-1} u + C = \sec^{-1}(e^x) + C$$

65. $\int \frac{x\,dx}{\sqrt{1-x^4}}$

SOLUTION Let $u = x^2$. Then $du = 2x\,dx$, and $\sqrt{1-x^4} = \sqrt{1-u^2}$. Thus,

$$\int \frac{x\,dx}{\sqrt{1-x^4}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u + C = \frac{1}{2}\sin^{-1}(x^2) + C.$$

66. $\int_0^1 \frac{dx}{25-x^2}$

SOLUTION Let $x = 5u$. Then $dx = 5\,du$, and the new limits of integration are $u = 0$ and $u = \frac{1}{5}$. Thus,

$$\int_0^1 \frac{dx}{25-x^2} = \frac{1}{25}\int_0^{1/5} \frac{5\,du}{1-u^2} = \frac{5}{25}\int_0^{1/5} \frac{du}{1-u^2}$$
$$= \frac{1}{5}\tanh^{-1}u\Big|_0^{1/5} = \frac{1}{5}\left(\tanh^{-1}\frac{1}{5} - \tanh^{-1}0\right) = \frac{1}{5}\tanh^{-1}\frac{1}{5}.$$

67. $\int_0^4 \frac{dx}{2x^2+1}$

SOLUTION Let $u = \sqrt{2}x$. Then $du = \sqrt{2}\,dx$, and the new limits of integration are $u = 0$ and $u = 4\sqrt{2}$. Thus,

$$\int_0^4 \frac{dx}{2x^2+1} = \int_0^{4\sqrt{2}} \frac{\frac{1}{\sqrt{2}}\,du}{u^2+1} = \frac{1}{\sqrt{2}}\int_0^{4\sqrt{2}} \frac{du}{u^2+1}$$
$$= \frac{1}{\sqrt{2}}\tan^{-1}u\Big|_0^{4\sqrt{2}} = \frac{1}{\sqrt{2}}\left(\tan^{-1}(4\sqrt{2}) - \tan^{-1}0\right) = \frac{1}{\sqrt{2}}\tan^{-1}(4\sqrt{2}).$$

68. $\int_5^8 \frac{dx}{x\sqrt{x^2-16}}$

SOLUTION Let $x = 4u$. Then $dx = 4\,du$, and the new limits of integration are $u = \frac{5}{4}$ and $u = 2$. Thus,

$$\int_5^8 \frac{dx}{x\sqrt{x^2-16}} = \frac{1}{4}\int_{5/4}^2 \frac{du}{u\sqrt{u^2-1}} = \frac{1}{4}\left(\sec^{-1}u\right)\Big|_{5/4}^2 = \frac{1}{4}\left(\sec^{-1}2 - \sec^{-1}\frac{5}{4}\right) = \frac{1}{4}\left(\frac{\pi}{3} - \sec^{-1}\frac{5}{4}\right).$$

69. $\int_0^1 \frac{(\tan^{-1}x)^3\,dx}{1+x^2}$

SOLUTION Let $u = \tan^{-1}x$. Then

$$du = \frac{1}{1+x^2}\,dx$$

and

$$\int_0^1 \frac{(\tan^{-1}x)^3\,dx}{1+x^2} = \int_0^{\pi/4} u^3\,du = \frac{1}{4}u^4\Big|_0^{\pi/4} = \frac{1}{4}\left(\frac{\pi}{4}\right)^4 = \frac{\pi^4}{1024}.$$

70. $\int \frac{\cos^{-1}t\,dt}{\sqrt{1-t^2}}$

SOLUTION Let $u = \cos^{-1}t$. Then $du = -\frac{1}{\sqrt{1-t^2}}\,dt$, and

$$\int \frac{\cos^{-1}t}{\sqrt{1-t^2}}\,dt = -\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}(\cos^{-1}t)^2 + C.$$

71. Combine to write as a single integral:

$$\int_0^8 f(x)\,dx + \int_{-2}^0 f(x)\,dx + \int_8^6 f(x)\,dx$$

SOLUTION First, rewrite

$$\int_0^8 f(x)\,dx = \int_0^6 f(x)\,dx + \int_6^8 f(x)\,dx$$

and observe that

$$\int_8^6 f(x)\,dx = -\int_6^8 f(x)\,dx.$$

Thus,

$$\int_0^8 f(x)\,dx + \int_8^6 f(x)\,dx = \int_0^6 f(x)\,dx.$$

Finally,

$$\int_0^8 f(x)\,dx + \int_{-2}^0 f(x)\,dx + \int_8^6 f(x)\,dx = \int_0^6 f(x)\,dx + \int_{-2}^0 f(x)\,dx = \int_{-2}^6 f(x)\,dx.$$

72. Let $A(x) = \int_0^x f(x)\,dx$, where $f(x)$ is the function shown in Figure 4. Identify the location of the local minima, the local maxima, and points of inflection of $A(x)$ on the interval $[0, E]$, as well as the intervals where $A(x)$ is increasing, decreasing, concave up, or concave down. Where does the absolute max of $A(x)$ occur?

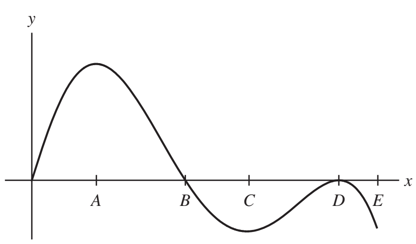


FIGURE 4

SOLUTION Let $f(x)$ be the function shown in Figure 4 and define

$$A(x) = \int_0^x f(x)\,dx.$$

Then $A'(x) = f(x)$ and $A''(x) = f'(x)$. Hence, $A(x)$ is increasing when $f(x)$ is positive, is decreasing when $f(x)$ is negative, is concave up when $f(x)$ is increasing and is concave down when $f(x)$ is decreasing. Thus, $A(x)$ is increasing for $0 < x < B$, is decreasing for $B < x < D$ and for $D < x < E$, has a local maximum at $x = B$ and no local minima. Moreover, $A(x)$ is concave up for $0 < x < A$ and for $C < x < D$, is concave down for $A < x < C$ and for $D < x < E$, and has a point of inflection at $x = A$, $x = C$ and $x = D$. The absolute maximum value for $A(x)$ occurs at $x = B$.

73. Find the local minima, the local maxima, and the inflection points of $A(x) = \int_3^x \frac{t\,dt}{t^2+1}$.

SOLUTION Let

$$A(x) = \int_3^x \frac{t\,dt}{t^2+1}.$$

Then

$$A'(x) = \frac{x}{x^2+1}$$

and

$$A''(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

Now, $x = 0$ is the only critical point of A; because $A''(0) > 0$, it follows that A has a local minimum at $x = 0$. There are no local maxima. Moreover, $A(x)$ is concave down for $|x| > 1$ and concave up for $|x| < 1$. $A(x)$ therefore has inflection points at $x = \pm 1$.

74. A particle starts at the origin at time $t = 0$ and moves with velocity $v(t)$ as shown in Figure 5.

(a) How many times does the particle return to the origin in the first 12 seconds?
(b) What is the particle's maximum distance from the origin?
(c) What is particle's maximum distance to the left of the origin?

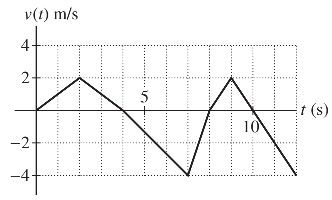


FIGURE 5

SOLUTION Because the particle starts at the origin, the position of the particle is given by

$$s(t) = \int_0^t v(\tau)\,d\tau;$$

that is by the signed area between the graph of the velocity and the t-axis over the interval $[0, t]$. Using the geometry in Figure 5, we see that $s(t)$ is increasing for $0 < t < 4$ and for $8 < t < 10$ and is decreasing for $4 < t < 8$ and for $10 < t < 12$. Furthermore,

$$s(0) = 0 \text{ m}, \quad s(4) = 4 \text{ m}, \quad s(8) = -4 \text{ m}, \quad s(10) = -2 \text{ m}, \quad \text{and} \quad s(12) = -6 \text{ m}.$$

(a) In the first 12 seconds, the particle returns to the origin once, sometime between $t = 4$ and $t = 8$ seconds.
(b) The particle's maximum distance from the origin is 6 meters (to the left at $t = 12$ seconds).
(c) The particle's distance to the left of the origin is 6 meters.

75. On a typical day, a city consumes water at the rate of $r(t) = 100 + 72t - 3t^2$ (in thousands of gallons per hour), where t is the number of hours past midnight. What is the daily water consumption? How much water is consumed between 6 PM and midnight?

SOLUTION With a consumption rate of $r(t) = 100 + 72t - 3t^2$ thousand gallons per hour, the daily consumption of water is

$$\int_0^{24} (100 + 72t - 3t^2)\,dt = \left(100t + 36t^2 - t^3\right)\Big|_0^{24} = 100(24) + 36(24)^2 - (24)^3 = 9312,$$

or 9.312 million gallons. From 6 PM to midnight, the water consumption is

$$\begin{aligned}\int_{18}^{24} (100 + 72t - 3t^2)\,dt &= \left(100t + 36t^2 - t^3\right)\Big|_{18}^{24}\\ &= 100(24) + 36(24)^2 - (24)^3 - \left(100(18) + 36(18)^2 - (18)^3\right)\\ &= 9312 - 7632 = 1680,\end{aligned}$$

or 1.68 million gallons.

76. The learning curve in a certain bicycle factory is $L(x) = 12x^{-1/5}$ (in hours per bicycle), which means that it takes a bike mechanic $L(n)$ hours to assemble the nth bicycle. If a mechanic has produced 24 bicycles, how long does it take her or him to produce the second batch of 12?

SOLUTION The second batch of 12 bicycles consists of bicycles 13 through 24. The time it takes to produce these bicycles is

$$\int_{13}^{24} 12x^{-1/5}\,dx = 15x^{4/5}\Big|_{13}^{24} = 15\left(24^{4/5} - 13^{4/5}\right) \approx 73.91 \text{ hours}.$$

77. Cost engineers at NASA have the task of projecting the cost P of major space projects. It has been found that the cost C of developing a projection increases with P at the rate $dC/dP \approx 21P^{-0.65}$, where C is in thousands of dollars and P in millions of dollars. What is the cost of developing a projection for a project whose cost turns out to be $P = \$35$ million?

SOLUTION Assuming it costs nothing to develop a projection for a project with a cost of \$0, the cost of developing a projection for a project whose cost turns out to be \$35 million is

$$\int_0^{35} 21P^{-0.65}\,dP = 60P^{0.35}\Big|_0^{35} = 60(35)^{0.35} \approx 208.245,$$

or \$208,245.

78. An astronomer estimates that in a certain constellation, the number of stars per magnitude m, per degree-squared of sky, is equal to $A(m) = 2.4 \times 10^{-6} m^{7.4}$ (fainter stars have higher magnitudes). Determine the total number of stars of magnitude between 6 and 15 in a one-degree-squared region of sky.

SOLUTION The total number of stars of magnitude between 6 and 15 in a one-degree-squared region of sky is

$$\int_6^{15} A(m)\,dm = \int_6^{15} 2.4 \times 10^{-6} m^{7.4}\,dm$$
$$= \frac{2}{7} \times 10^{-6} m^{8.4} \Big|_6^{15}$$
$$\approx 2162$$

79. Evaluate $\displaystyle\int_{-8}^{8} \frac{x^{15}\,dx}{3 + \cos^2 x}$, using the properties of odd functions.

SOLUTION Let $f(x) = \frac{x^{15}}{3+\cos^2 x}$ and note that

$$f(-x) = \frac{(-x)^{15}}{3 + \cos^2(-x)} = -\frac{x^{15}}{\cos^2 x} = -f(x).$$

Because $f(x)$ is an odd function and the interval $-8 \le x \le 8$ is symmetric about $x = 0$, it follows that

$$\int_{-8}^{8} \frac{x^{15}\,dx}{3 + \cos^2 x} = 0.$$

80. Evaluate $\int_0^1 f(x)\,dx$, assuming that $f(x)$ is an even continuous function such that

$$\int_1^2 f(x)\,dx = 5, \qquad \int_{-2}^{1} f(x)\,dx = 8$$

SOLUTION Using the given information

$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{1} f(x)\,dx + \int_1^2 f(x)\,dx = 13.$$

Because $f(x)$ is an even function, it follows that

$$\int_{-2}^{0} f(x)\,dx = \int_0^2 f(x)\,dx,$$

so

$$\int_0^2 f(x)\,dx = \frac{13}{2}.$$

Finally,

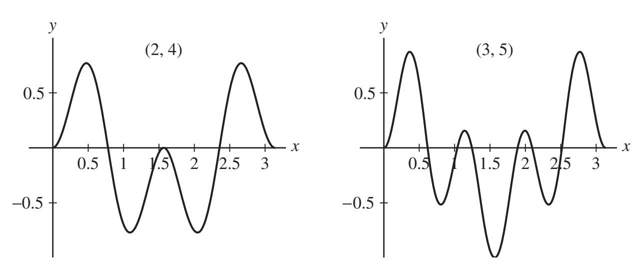
$$\int_0^1 f(x)\,dx = \int_0^2 f(x)\,dx - \int_1^2 f(x)\,dx = \frac{13}{2} - 5 = \frac{3}{2}.$$

81. GU Plot the graph of $f(x) = \sin mx \sin nx$ on $[0, \pi]$ for the pairs $(m, n) = (2, 4), (3, 5)$ and in each case guess the value of $I = \int_0^\pi f(x)\,dx$. Experiment with a few more values (including two cases with $m = n$) and formulate a conjecture for when I is zero.

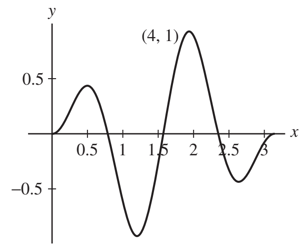
SOLUTION The graphs of $f(x) = \sin mx \sin nx$ with $(m, n) = (2, 4)$ and $(m, n) = (3, 5)$ are shown below. It appears as if the positive areas balance the negative areas, so we expect that

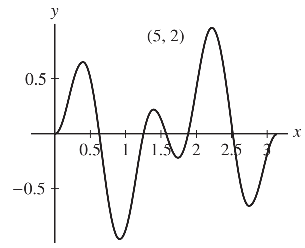
$$I = \int_0^\pi f(x)\,dx = 0$$

in these cases.



We arrive at the same conclusion for the cases $(m, n) = (4, 1)$ and $(m, n) = (5, 2)$.

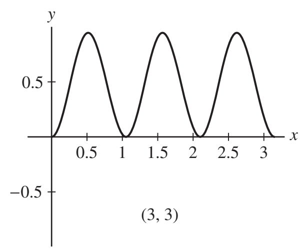


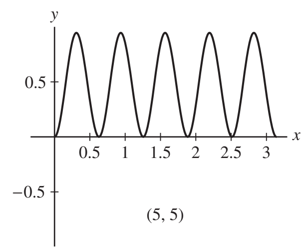


However, when $(m, n) = (3, 3)$ and when $(m, n) = (5, 5)$, the value of

$$I = \int_0^{\pi} f(x)\, dx$$

is clearly not zero as there is no negative area.





We therefore conjecture that I is zero whenever $m \neq n$.

82. Show that

$$\int x\, f(x)\, dx = xF(x) - G(x)$$

where $F'(x) = f(x)$ and $G'(x) = F(x)$. Use this to evaluate $\int x \cos x\, dx$.

SOLUTION Suppose $F'(x) = f(x)$ and $G'(x) = F(x)$. Then

$$\frac{d}{dx}(xF(x) - G(x)) = xF'(x) + F(x) - G'(x) = xf(x) + F(x) - F(x) = xf(x).$$

Therefore, $xF(x) - G(x)$ is an antiderivative of $xf(x)$ and

$$\int xf(x)\, dx = xF(x) - G(x) + C.$$

To evaluate $\int x \cos x\, dx$, note that $f(x) = \cos x$. Thus, we may take $F(x) = \sin x$ and $G(x) = -\cos x$. Finally,

$$\int x \cos x\, dx = x \sin x + \cos x + C.$$

83. Prove

$$2 \le \int_1^2 2^x\, dx \le 4 \qquad \text{and} \qquad \frac{1}{9} \le \int_1^2 3^{-x}\, dx \le \frac{1}{3}$$

SOLUTION The function $f(x) = 2^x$ is increasing, so $1 \le x \le 2$ implies that $2 = 2^1 \le 2^x \le 2^2 = 4$. Consequently,

$$2 = \int_1^2 2\, dx \le \int_1^2 2^x\, dx \le \int_1^2 4\, dx = 4.$$

On the other hand, the function $f(x) = 3^{-x}$ is decreasing, so $1 \le x \le 2$ implies that

$$\frac{1}{9} = 3^{-2} \le 3^{-x} \le 3^{-1} = \frac{1}{3}.$$

It then follows that

$$\frac{1}{9} = \int_1^2 \frac{1}{9}\, dx \le \int_1^2 3^{-x}\, dx \le \int_1^2 \frac{1}{3}\, dx = \frac{1}{3}.$$

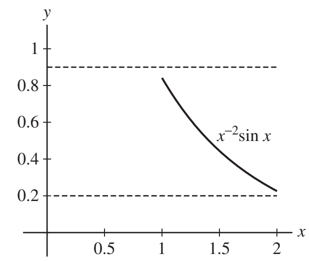
84. GU Plot the graph of $f(x) = x^{-2}\sin x$, and show that $0.2 \le \int_1^2 f(x)\,dx \le 0.9$.

SOLUTION Let $f(x) = x^{-2}\sin x$. From the figure below, we see that

$$0.2 \le f(x) \le 0.9$$

for $1 \le x \le 2$. Therefore,

$$0.2 = \int_0^1 0.2\,dx \le \int_0^1 f(x)\,dx \le \int_0^1 0.9\,dx = 0.9.$$



85. Find upper and lower bounds for $\int_0^1 f(x)\,dx$, for $f(x)$ in Figure 6.

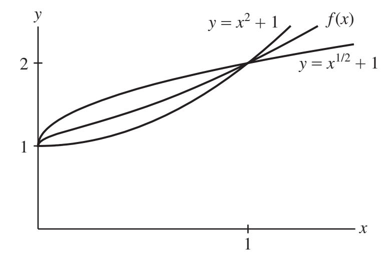


FIGURE 6

SOLUTION From the figure, we see that the inequalities $x^2 + 1 \le f(x) \le \sqrt{x} + 1$ hold for $0 \le x \le 1$. Because

$$\int_0^1 (x^2+1)\,dx = \left(\frac{1}{3}x^3 + x\right)\Big|_0^1 = \frac{4}{3}$$

and

$$\int_0^1 (\sqrt{x}+1)\,dx = \left(\frac{2}{3}x^{3/2} + x\right)\Big|_0^1 = \frac{5}{3},$$

it follows that

$$\frac{4}{3} \le \int_0^1 f(x)\,dx \le \frac{5}{3}.$$

In Exercises 86–91, find the derivative.

86. $A'(x)$, where $A(x) = \int_3^x \sin(t^3)\,dt$

SOLUTION Let $A(x) = \int_3^x \sin(t^3)\,dt$. Then $A'(x) = \sin(x^3)$.

87. $A'(\pi)$, where $A(x) = \int_2^x \frac{\cos t}{1+t}\,dt$

SOLUTION Let $A(x) = \int_2^x \frac{\cos t}{1+t}\,dt$. Then $A'(x) = \frac{\cos x}{1+x}$ and

$$A'(\pi) = \frac{\cos\pi}{1+\pi} = -\frac{1}{1+\pi}.$$

88. $\frac{d}{dy}\int_{-2}^{y} 3^x\,dx$

SOLUTION $\frac{d}{dy}\int_{-2}^{y} 3^x\,dx = 3^y$.

89. $G'(x)$, where $G(x) = \int_{-2}^{\sin x} t^3\,dt$

SOLUTION Let $G(x) = \int_{-2}^{\sin x} t^3\,dt$. Then

$$G'(x) = \sin^3 x \frac{d}{dx} \sin x = \sin^3 x \cos x.$$

90. $G'(2)$, where $G(x) = \int_0^{x^3} \sqrt{t+1}\,dt$

SOLUTION Let $G(x) = \int_0^{x^3} \sqrt{t+1}\,dt$. Then

$$G'(x) = \sqrt{x^3+1}\frac{d}{dx}x^3 = 3x^2\sqrt{x^3+1}$$

and $G'(2) = 3(2)^2\sqrt{8+1} = 36$.

91. $H'(1)$, where $H(x) = \int_{4x^2}^{9} \frac{1}{t}\,dt$

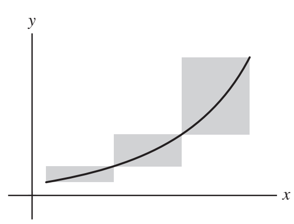
SOLUTION Let $H(x) = \int_{4x^2}^{9} \frac{1}{t}\,dt = -\int_9^{4x^2} \frac{1}{t}\,dt$. Then

$$H'(x) = -\frac{1}{4x^2}\frac{d}{dx}4x^2 = -\frac{8x}{4x^2} = -\frac{2}{x}$$

and $H'(1) = -2$.

92. Explain with a graph: If $f(x)$ is increasing and concave up on $[a, b]$, then L_N is more accurate than R_N. Which is more accurate if $f(x)$ is increasing and concave down?

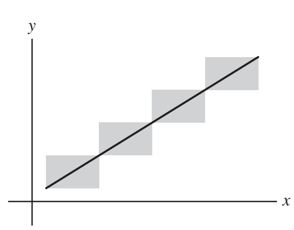
SOLUTION Consider the figure below, which displays a portion of the graph of an increasing, concave up function.



The shaded rectangles represent the differences between the right-endpoint approximation R_N and the left-endpoint approximation L_N. In particular, the portion of each rectangle that lies below the graph of $y = f(x)$ is the amount by which L_N underestimates the area under the graph, whereas the portion of each rectangle that lies above the graph of $y = f(x)$ is the amount by which R_N overestimates the area. Because the graph of $y = f(x)$ is increasing and concave up, the lower portion of each shaded rectangle is smaller than the upper portion. Therefore, L_N is more accurate (introduces less error) than R_N. By similar reasoning, if $f(x)$ is increasing and concave down, then R_N is more accurate than L_N.

93. Explain with a graph: If $f(x)$ is linear on $[a, b]$, then the $\int_a^b f(x)\,dx = \frac{1}{2}(R_N + L_N)$ for all N.

SOLUTION Consider the figure below, which displays a portion of the graph of a linear function.



The shaded rectangles represent the differences between the right-endpoint approximation R_N and the left-endpoint approximation L_N. In particular, the portion of each rectangle that lies below the graph of $y = f(x)$ is the amount by which L_N underestimates the area under the graph, whereas the portion of each rectangle that lies above the graph of $y = f(x)$ is the amount by which R_N overestimates the area. Because the graph of $y = f(x)$ is a line, the lower portion of each shaded rectangle is exactly the same size as the upper portion. Therefore, if we average L_N and R_N, the error in the two approximations will exactly cancel, leaving

$$\frac{1}{2}(R_N + L_N) = \int_a^b f(x)\,dx.$$

94. In this exercise, we prove

$$x - \frac{x^2}{2} \le \ln(1+x) \le x \qquad (\text{for } x > 0) \qquad \boxed{1}$$

(a) Show that $\ln(1+x) = \int_0^x \frac{dt}{1+t}$ for $x > 0$.

(b) Verify that $1 - t \le \frac{1}{1+t} \le 1$ for all $t > 0$.

(c) Use (b) to prove Eq. (1).

(d) Verify Eq. (1) for $x = 0.5$, 0.1, and 0.01.

SOLUTION

(a) Let $x > 0$. Then

$$\int_0^x \frac{dt}{1+t} = \ln(1+t)\Big|_0^x = \ln(1+x) - \ln 1 = \ln(1+x).$$

(b) For $t > 0$, $1 + t > 1$, so $\frac{1}{1+t} < 1$. Moreover, $(1-t)(1+t) = 1 - t^2 < 1$. Because $1 + t > 0$, it follows that $1 - t < \frac{1}{1+t}$. Hence,

$$1 - t \le \frac{1}{1+t} \le 1.$$

(c) Integrating each expression in the result from part (b) from $t = 0$ to $t = x$ yields

$$x - \frac{x^2}{2} \le \ln(1+x) \le x.$$

(d) For $x = 0.5$, $x = 0.1$ and $x = 0.01$, we obtain the string of inequalities

$$\begin{aligned} 0.375 \le&\ 0.405465 \le 0.5 \\ 0.095 \le&\ 0.095310 \le 0.1 \\ 0.00995 \le&\ 0.00995033 \le 0.01, \end{aligned}$$

respectively.

95. Let

$$F(x) = x\sqrt{x^2-1} - 2\int_1^x \sqrt{t^2-1}\,dt$$

Prove that $F(x)$ and $\cosh^{-1} x$ differ by a constant by showing that they have the same derivative. Then prove they are equal by evaluating both at $x = 1$.

SOLUTION Let

$$F(x) = x\sqrt{x^2-1} - 2\int_1^x \sqrt{t^2-1}\,dt.$$

Then

$$\frac{dF}{dx} = \sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}} - 2\sqrt{x^2-1} = \frac{x^2}{\sqrt{x^2-1}} - \sqrt{x^2-1} = \frac{1}{\sqrt{x^2-1}}.$$

Also, $\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$; therefore, $F(x)$ and $\cosh^{-1}x$ have the same derivative. We conclude that $F(x)$ and $\cosh^{-1}x$ differ by a constant:

$$F(x) = \cosh^{-1}x + C.$$

Now, let $x = 1$. Because $F(1) = 0$ and $\cosh^{-1} 1 = 0$, it follows that $C = 0$. Therefore,

$$F(x) = \cosh^{-1}x.$$

96. Let $f(x)$ be a positive increasing continuous function on $[a, b]$, where $0 \le a < b$ as in Figure 7. Show that the shaded region has area

$$I = bf(b) - af(a) - \int_a^b f(x)\,dx \qquad \boxed{2}$$

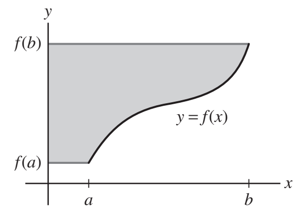


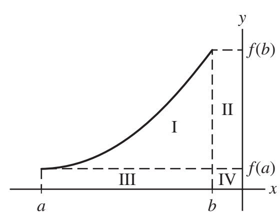
FIGURE 7

SOLUTION We can construct the shaded region in Figure 7 by taking a rectangle of length b and height $f(b)$ and removing a rectangle of length a and height $f(a)$ as well as the region between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$. The area of the resulting region is then the area of the large rectangle minus the area of the small rectangle and minus the area under the curve $y = f(x)$; that is,

$$I = bf(b) - af(a) - \int_a^b f(x)\,dx.$$

97. How can we interpret the quantity I in Eq. (2) if $a < b \le 0$? Explain with a graph.

SOLUTION We will consider each term on the right-hand side of (2) separately. For convenience, let **I**, **II**, **III** and **IV** denote the area of the similarly labeled region in the diagram below.



Because $b < 0$, the expression $bf(b)$ is the opposite of the area of the rectangle along the right; that is,

$$bf(b) = -\mathbf{II} - \mathbf{IV}.$$

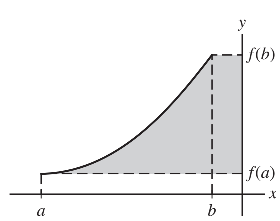
Similarly,

$$-af(a) = \mathbf{III} + \mathbf{IV} \qquad \text{and} \qquad -\int_a^b f(x)\,dx = -\mathbf{I} - \mathbf{III}.$$

Therefore,

$$bf(b) - af(a) - \int_a^b f(x)\,dx = -\mathbf{I} - \mathbf{II};$$

that is, the opposite of the area of the shaded region shown below.



98. The isotope thorium-234 has a half-life of 24.5 days.

(a) What is the differential equation satisfied by $y(t)$, the amount of thorium-234 in a sample at time t?

(b) At $t = 0$, a sample contains 2 kg of thorium-234. How much remains after 40 days?

SOLUTION

(a) By the equation for half-life,

$$24.5 = \frac{\ln 2}{k}, \quad \text{so} \quad k = \frac{\ln 2}{24.5} \approx 0.028 \text{ days}^{-1}.$$

Therefore, the differential equation for $y(t)$ is

$$y' = -0.028y.$$

(b) If there are 2 kg of thorium-234 at $t = 0$, then $y(t) = 2e^{-0.028t}$. After 40 days, the amount of thorium-234 is

$$y(40) = 2e^{-0.028(40)} = 0.653 \text{ kg}.$$

99. The Oldest Snack Food? In Bat Cave, New Mexico, archaeologists found ancient human remains, including cobs of popping corn whose C^{14}-to-C^{12} ratio was approximately 48% of that found in living matter. Estimate the age of the corn cobs.

SOLUTION Let t be the age of the corn cobs. The C^{14} to C^{12} ratio decreased by a factor of $e^{-0.000121t}$ which is equal to 0.48. That is:

$$e^{-0.000121t} = 0.48,$$

so

$$-0.000121t = \ln 0.48,$$

and

$$t = -\frac{1}{0.000121} \ln 0.48 \approx 6065.9.$$

We conclude that the age of the corn cobs is approximately 6065.9 years.

100. The C^{14}-to-C^{12} ratio of a sample is proportional to the disintegration rate (number of beta particles emitted per minute) that is measured directly with a Geiger counter. The disintegration rate of carbon in a living organism is 15.3 beta particles per minute per gram. Find the age of a sample that emits 9.5 beta particles per minute per gram.

SOLUTION Let t be the age of the sample in years. Because the disintegration rate for the sample has dropped from 15.3 beta particles/min per gram to 9.5 beta particles/min per gram and the C^{14} to C^{12} ratio is proportional to the disintegration rate, it follows that

$$e^{-0.000121t} = \frac{9.5}{15.3},$$

so

$$t = -\frac{1}{0.000121} \ln \frac{9.5}{15.3} \approx 3938.5.$$

We conclude that the sample is approximately 3938.5 years old.

101. What is the interest rate if the PV of \$50,000 to be delivered in 3 years is \$43,000?

SOLUTION Let r denote the interest rate. The present value of \$50,000 received in 3 years with an interest rate of r is $50{,}000e^{-3r}$. Thus, we need to solve

$$43{,}000 = 50{,}000e^{-3r}$$

for r. This yields

$$r = -\frac{1}{3} \ln \frac{43}{50} = 0.0503.$$

Thus, the interest rate is 5.03%.

102. An equipment upgrade costing \$1 million will save a company \$320,000 per year for 4 years. Is this a good investment if the interest rate is $r = 5\%$? What is the largest interest rate that would make the investment worthwhile? Assume that the savings are received as a lump sum at the end of each year.

SOLUTION With an interest rate of $r = 5\%$, the present value of the four payments is

$$\$320{,}000\left(e^{-0.05} + e^{-0.1} + e^{-0.15} + e^{-0.2}\right) = \$1{,}131{,}361.78.$$

As this is greater than the \$1 million cost of the upgrade, this is a good investment. To determine the largest interest rate that would make the investment worthwhile, we must solve the equation

$$320{,}000\left(e^{-r} + e^{-2r} + e^{-3r} + e^{-4r}\right) = 1{,}000{,}000$$

for r. Using a computer algebra system, we find $r = 10.13\%$.

103. Find the PV of an income stream paying out continuously at a rate of $5000e^{-0.1t}$ dollars per year for 5 years, assuming an interest rate of $r = 4\%$.

SOLUTION $PV = \int_0^5 5000e^{-0.1t}e^{-0.04t}\,dt = \int_0^5 5000e^{-0.14t}\,dt = \frac{5000}{-0.14}e^{-0.14t}\Big|_0^5 = \$17,979.10.$

104. Calculate the limit:

(a) $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n$ **(b)** $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{4n}$ **(c)** $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^{3n}$

SOLUTION

(a) $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{1}{n/4}\right)^{n/4}\right]^4 = e^4.$

(b) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{4n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^4 = e^4.$

(c) $\lim_{n\to\infty}\left(1+\frac{4}{n}\right)^{3n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n/4}\right)^{n/4}\right]^{12} = e^{12}.$