2 | LIMITS

2.1 Limits, Rates of Change, and Tangent Lines

Preliminary Questions

1. Average velocity is equal to the slope of a secant line through two points on a graph. Which graph?

SOLUTION Average velocity is the slope of a secant line through two points on the graph of position as a function of time.

2. Can instantaneous velocity be defined as a ratio? If not, how is instantaneous velocity computed?

SOLUTION Instantaneous velocity cannot be defined as a ratio. It is defined as the limit of average velocity as time elapsed shrinks to zero.

3. What is the graphical interpretation of instantaneous velocity at a moment $t = t_0$?

SOLUTION Instantaneous velocity at time $t = t_0$ is the slope of the line tangent to the graph of position as a function of time at $t = t_0$.

4. What is the graphical interpretation of the following statement? The average rate of change approaches the instantaneous rate of change as the interval $[x_0, x_1]$ shrinks to x_0.

SOLUTION The slope of the secant line over the interval $[x_0, x_1]$ approaches the slope of the tangent line at $x = x_0$.

5. The rate of change of atmospheric temperature with respect to altitude is equal to the slope of the tangent line to a graph. Which graph? What are possible units for this rate?

SOLUTION The rate of change of atmospheric temperature with respect to altitude is the slope of the line tangent to the graph of atmospheric temperature as a function of altitude. Possible units for this rate of change are °F/ft or °C/m.

Exercises

1. A ball dropped from a state of rest at time $t = 0$ travels a distance $s(t) = 4.9t^2$ m in t seconds.

(a) How far does the ball travel during the time interval $[2, 2.5]$?

(b) Compute the average velocity over $[2, 2.5]$.

(c) Compute the average velocity for the time intervals in the table and estimate the ball's instantaneous velocity at $t = 2$.

Interval	$[2, 2.01]$	$[2, 2.005]$	$[2, 2.001]$	$[2, 2.00001]$
Average velocity				

SOLUTION

(a) During the time interval $[2, 2.5]$, the ball travels $\Delta s = s(2.5) - s(2) = 4.9(2.5)^2 - 4.9(2)^2 = 11.025$ m.

(b) The average velocity over $[2, 2.5]$ is

$$\frac{\Delta s}{\Delta t} = \frac{s(2.5) - s(2)}{2.5 - 2} = \frac{11.025}{0.5} = 22.05 \text{ m/s}.$$

(c)

time interval	$[2, 2.01]$	$[2, 2.005]$	$[2, 2.001]$	$[2, 2.00001]$
average velocity	19.649	19.6245	19.6049	19.600049

The instantaneous velocity at $t = 2$ is 19.6 m/s.

2. A wrench released from a state of rest at time $t = 0$ travels a distance $s(t) = 4.9t^2$ m in t seconds. Estimate the instantaneous velocity at $t = 3$.

SOLUTION To find the instantaneous velocity, we compute the average velocities:

time interval	$[3, 3.01]$	$[3, 3.005]$	$[3, 3.001]$	$[3, 3.00001]$
average velocity	29.449	29.4245	29.4049	29.400049

The instantaneous velocity is approximately 29.4 m/s.

3. Let $v = 20\sqrt{T}$ as in Example 2. Estimate the instantaneous rate of change of v with respect to T when $T = 300$ K.

SOLUTION

T interval	[300, 300.01]	[300, 300.005]
average rate of change	0.577345	0.577348
T interval	[300, 300.001]	[300, 300.00001]
average rate of change	0.57735	0.57735

The instantaneous rate of change is approximately 0.57735 $\text{m}/(\text{s}\cdot\text{K})$.

4. Compute $\Delta y/\Delta x$ for the interval [2, 5], where $y = 4x - 9$. What is the instantaneous rate of change of y with respect to x at $x = 2$?

SOLUTION $\Delta y/\Delta x = ((4(5) - 9) - (4(2) - 9))/(5 - 2) = 4$. Because the graph of $y = 4x - 9$ is a line with slope 4, the average rate of change of y calculated over any interval will be equal to 4; hence, the instantaneous rate of change at any x will also be equal to 4.

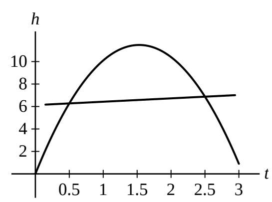
In Exercises 5 and 6, a stone is tossed vertically into the air from ground level with an initial velocity of 15 m/s. *Its height at time t is* $h(t) = 15t - 4.9t^2$ m.

5. Compute the stone's average velocity over the time interval [0.5, 2.5] and indicate the corresponding secant line on a sketch of the graph of $h(t)$.

SOLUTION The average velocity is equal to

$$\frac{h(2.5) - h(0.5)}{2} = 0.3.$$

The secant line is plotted with $h(t)$ below.



6. Compute the stone's average velocity over the time intervals [1, 1.01], [1, 1.001], [1, 1.0001] and [0.99, 1], [0.999, 1], [0.9999, 1], and then estimate the instantaneous velocity at $t = 1$.

SOLUTION With $h(t) = 15t - 4.9t^2$, the average velocity over the time interval $[t_1, t_2]$ is given by

$$\frac{\Delta h}{\Delta t} = \frac{h(t_2) - h(t_1)}{t_2 - t_1}.$$

time interval	[1, 1.01]	[1, 1.001]	[1, 1.0001]	[0.99, 1]	[0.999, 1]	[0.9999, 1]
average velocity	5.151	5.1951	5.1995	5.249	5.2049	5.2005

The instantaneous velocity at $t = 1$ second is 5.2 m/s.

7. With an initial deposit of $100, the balance in a bank account after t years is $f(t) = 100(1.08)^t$ dollars.

(a) What are the units of the rate of change of $f(t)$?

(b) Find the average rate of change over [0, 0.5] and [0, 1].

(c) Estimate the instantaneous rate of change at $t = 0.5$ by computing the average rate of change over intervals to the left and right of $t = 0.5$.

SOLUTION

(a) The units of the rate of change of $f(t)$ are dollars/year or $/yr.

(b) The average rate of change of $f(t) = 100(1.08)^t$ over the time interval $[t_1, t_2]$ is given by

$$\frac{\Delta f}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

time interval	[0, .5]	[0, 1]
average rate of change	7.8461	8

(c)

time interval	[0.5, 0.51]	[0.5, 0.501]	[0.5, 0.5001]
average rate of change	8.0011	7.9983	7.9981
time interval	[0.49, 0.5]	[0.499, 0.5]	[0.4999, 0.5]
average rate of change	7.9949	7.9977	7.998

The rate of change at $t = 0.5$ is approximately \$8/yr.

8. The position of a particle at time t is $s(t) = t^3 + t$. Compute the average velocity over the time interval [1, 4] and estimate the instantaneous velocity at $t = 1$.

SOLUTION The average velocity over the time interval [1, 4] is

$$\frac{s(4) - s(1)}{4 - 1} = \frac{68 - 2}{3} = 22.$$

To estimate the instantaneous velocity at $t = 1$, we examine the following table.

time interval	[1, 1.01]	[1, 1.001]	[1, 1.0001]	[0.99, 1]	[0.999, 1]	[0.9999, 1]
average rate of change	4.0301	4.0030	4.0003	3.9701	3.9970	3.9997

The rate of change at $t = 1$ is approximately 4.

9. Figure 1 shows the estimated number N of Internet users in Chile, based on data from the United Nations Statistics Division.

(a) Estimate the rate of change of N at $t = 2003.5$.

(b) Does the rate of change increase or decrease as t increases? Explain graphically.

(c) Let R be the average rate of change over [2001, 2005]. Compute R.

(d) Is the rate of change at $t = 2002$ greater than or less than the average rate R? Explain graphically.

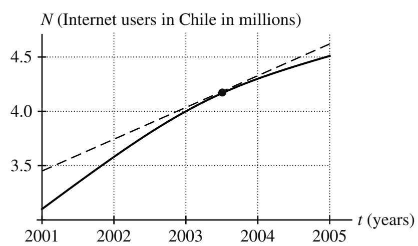


FIGURE 1

SOLUTION

(a) The tangent line shown in Figure 1 appears to pass through the points (2002, 3.75) and (2005, 4.6). Thus, the rate of change of N at $t = 2003.5$ is approximately

$$\frac{4.6 - 3.75}{2005 - 2002} = 0.283$$

million Internet users per year.

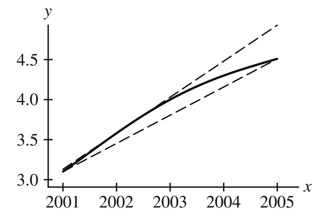
(b) As t increases, we move from left to right along the graph in Figure 1. Moreover, as we move from left to right along the graph, the slope of the tangent line decreases. Thus, the rate of change decreases as t increases.

(c) The graph of $N(t)$ appear to pass through the points (2001, 3.1) and (2005, 4.5). Thus, the average rate of change over [2001, 2005] is approximately

$$R = \frac{4.5 - 3.1}{2005 - 2001} = 0.35$$

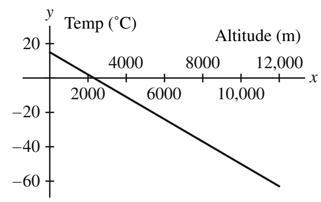
million Internet users per year.

(d) For the figure below, we see that the slope of the tangent line at $t = 2002$ is larger than the slope of the secant line through the endpoints of the graph of $N(t)$. Thus, the rate of change at $t = 2002$ is greater than the average rate of change R.



10. The **atmospheric temperature** T (in °C) at altitude h meters above a certain point on earth is $T = 15 - 0.0065h$ for $h \le 12{,}000$ m. What are the average and instantaneous rates of change of T with respect to h? Why are they the same? Sketch the graph of T for $h \le 12{,}000$.

SOLUTION The average and instantaneous rates of change of T with respect to h are both -0.0065°C/m. The rates of change are the same because T is a linear function of h with slope -0.0065.



In Exercises 11–18, estimate the instantaneous rate of change at the point indicated.

11. $P(x) = 3x^2 - 5; \quad x = 2$

SOLUTION

x interval	[2, 2.01]	[2, 2.001]	[2, 2.0001]	[1.99, 2]	[1.999, 2]	[1.9999, 2]
average rate of change	12.03	12.003	12.0003	11.97	11.997	11.9997

The rate of change at $x = 2$ is approximately 12.

12. $f(t) = 12t - 7; \quad t = -4$

SOLUTION

t interval	[−4, −3.99]	[−4, −3.999]	[−4, −3.9999]
average rate of change	12	12	12
t interval	[−4.01, −4]	[−4.001, −4]	[−4.0001, −4]
average rate of change	12	12	12

The rate of change at $t = -4$ is 12, as the graph of $y = f(t)$ is a line with slope 12.

13. $y(x) = \dfrac{1}{x+2}; \quad x = 2$

SOLUTION

x interval	[2, 2.01]	[2, 2.001]	[2, 2.0001]	[1.99, 2]	[1.999, 2]	[1.9999, 2]
average rate of change	−0.0623	−0.0625	−0.0625	−0.0627	−0.0625	−0.0625

The rate of change at $x = 2$ is approximately -0.06.

14. $y(t) = \sqrt{3t+1}; \quad t = 1$

SOLUTION

t interval	[1, 1.01]	[1, 1.001]	[1, 1.0001]	[0.99, 1]	[0.999, 1]	[0.9999, 1]
average rate of change	0.7486	0.7499	0.7500	0.7514	0.7501	0.7500

The rate of change at $t = 1$ is approximately 0.75.

15. $f(x) = e^x; \quad x = 0$

SOLUTION

x interval	$[-0.01, 0]$	$[-0.001, 0]$	$[-0.0001, 0]$	$[0, 0.01]$	$[0, 0.001]$	$[0, 0.0001]$
average rate of change	0.9950	0.9995	0.99995	1.0050	1.0005	1.00005

The rate of change at $x = 0$ is approximately 1.00.

16. $f(x) = e^x; \quad x = e$

SOLUTION

x interval	$[e - 0.01, e]$	$[e - 0.001, e]$	$[e - 0.0001, e]$	$[e, e + 0.01]$	$[e, e + 0.001]$	$[e, e + 0.0001]$
average rate of change	15.0787	15.1467	15.1535	15.2303	15.1618	15.1550

The rate of change at $x = e$ is approximately 15.15.

17. $f(x) = \ln x; \quad x = 3$

SOLUTION

x interval	$[2.99, 3]$	$[2.999, 3]$	$[2.9999, 3]$	$[3, 3.01]$	$[3, 3.001]$	$[3, 3.0001]$
average rate of change	0.33389	0.33339	0.33334	0.33278	0.33328	0.33333

The rate of change at $x = 3$ is approximately 0.333.

18. $f(x) = \tan^{-1} x; \quad x = \dfrac{\pi}{4}$

SOLUTION

x interval	$\left[\frac{\pi}{4} - 0.01, \frac{\pi}{4}\right]$	$\left[\frac{\pi}{4} - 0.001, \frac{\pi}{4}\right]$	$\left[\frac{\pi}{4} - 0.0001, \frac{\pi}{4}\right]$	$\left[\frac{\pi}{4}, \frac{\pi}{4} + 0.01\right]$	$\left[\frac{\pi}{4}, \frac{\pi}{4} + 0.001\right]$	$\left[\frac{\pi}{4}, \frac{\pi}{4} + 0.0001\right]$
average rate of change	0.6215	0.6188	0.6185	0.6155	0.6182	0.6185

The rate of change at $x = \frac{\pi}{4}$ is approximately 0.619.

19. The height (in centimeters) at time t (in seconds) of a small mass oscillating at the end of a spring is $h(t) = 8\cos(12\pi t)$.

(a) Calculate the mass's average velocity over the time intervals $[0, 0.1]$ and $[3, 3.5]$.

(b) Estimate its instantaneous velocity at $t = 3$.

SOLUTION

(a) The average velocity over the time interval $[t_1, t_2]$ is given by $\dfrac{\Delta h}{\Delta t} = \dfrac{h(t_2) - h(t_1)}{t_2 - t_1}$.

time interval	$[0, 0.1]$	$[3, 3.5]$
average velocity	-144.721 cm/s	0 cm/s

(b)

time interval	$[3, 3.0001]$	$[3, 3.00001]$	$[3, 3.000001]$	$[2.9999, 3]$	$[2.99999, 3]$	$[2.999999, 3]$
average velocity	-0.5685	-0.05685	-0.005685	0.5685	0.05685	0.005685

The instantaneous velocity at $t = 3$ seconds is approximately 0 cm/s.

20. The number $P(t)$ of *E. coli* cells at time t (hours) in a petri dish is plotted in Figure 2.

(a) Calculate the average rate of change of $P(t)$ over the time interval $[1, 3]$ and draw the corresponding secant line.

(b) Estimate the slope m of the line in Figure 2. What does m represent?

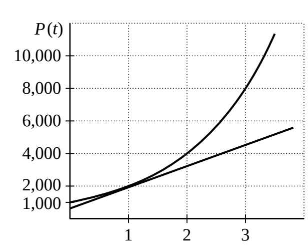


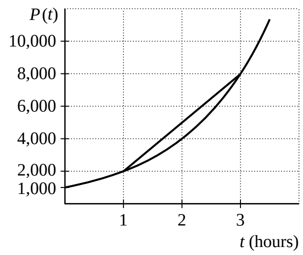
FIGURE 2 Number of *E. coli* cells at time t.

SOLUTION

(a) Looking at the graph, we can estimate $P(1) = 2000$ and $P(3) = 8000$. Assuming these values of $P(t)$, the average rate of change is

$$\frac{P(3) - P(1)}{3 - 1} = \frac{6000}{2} = 3000 \text{ cells/hour.}$$

The secant line is here:



(b) The line in Figure 2 goes through two points with approximate coordinates $(1, 2000)$ and $(2.5, 4000)$. This line has approximate slope

$$m = \frac{4000 - 2000}{2.5 - 1} = \frac{4000}{3} \text{ cells/hour.}$$

m is close to the slope of the line tangent to the graph of $P(t)$ at $t = 1$, and so m represents the instantaneous rate of change of $P(t)$ at $t = 1$ hour.

21.  Assume that the period T (in seconds) of a pendulum (the time required for a complete back-and-forth cycle) is $T = \frac{3}{2}\sqrt{L}$, where L is the pendulum's length (in meters).

(a) What are the units for the rate of change of T with respect to L? Explain what this rate measures.

(b) Which quantities are represented by the slopes of lines A and B in Figure 3?

(c) Estimate the instantaneous rate of change of T with respect to L when $L = 3$ m.

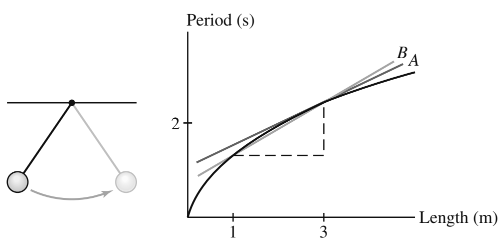


FIGURE 3 The period T is the time required for a pendulum to swing back and forth.

SOLUTION

(a) The units for the rate of change of T with respect to L are seconds per meter. This rate measures the sensitivity of the period of the pendulum to a change in the length of the pendulum.

(b) The slope of the line B represents the average rate of change in T from $L = 1$ m to $L = 3$ m. The slope of the line A represents the instantaneous rate of change of T at $L = 3$ m.

(c)

time interval	[3, 3.01]	[3, 3.001]	[3, 3.0001]	[2.99, 3]	[2.999, 3]	[2.9999, 3]
average velocity	0.4327	0.4330	0.4330	0.4334	0.4330	0.4330

The instantaneous rate of change at $L = 1$ m is approximately 0.4330 s/m.

22. The graphs in Figure 4 represent the positions of moving particles as functions of time.

(a) Do the instantaneous velocities at times t_1, t_2, t_3 in (A) form an increasing or a decreasing sequence?

(b) Is the particle speeding up or slowing down in (A)?

(c) Is the particle speeding up or slowing down in (B)?

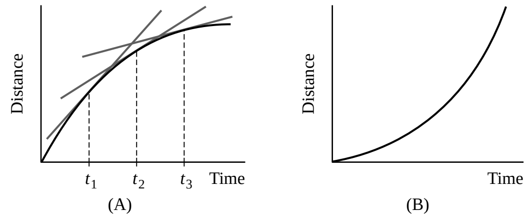


FIGURE 4

SOLUTION

(a) As the value of the independent variable increases, we note that the slope of the tangent lines decreases. Since Figure 4(A) displays position as a function of time, the slope of each tangent line is equal to the velocity of the particle; consequently, the velocities at t_1, t_2, t_3 form a decreasing sequence.

(b) Based on the solution to part (a), the velocity of the particle is decreasing; hence, the particle is slowing down.

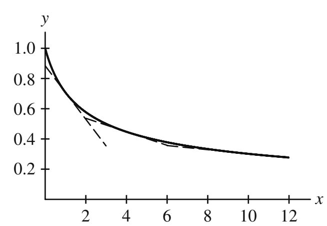
(c) If we were to draw several lines tangent to the graph in Figure 4(B), we would find that the slopes would be increasing. Accordingly, the velocity of the particle associated with Figure 4(B) is increasing, and the particle is speeding up.

23. GU An advertising campaign boosted sales of Crunchy Crust frozen pizza to a peak level of S_0 dollars per month. A marketing study showed that after t months, monthly sales declined to

$$S(t) = S_0 g(t), \quad \text{where } g(t) = \frac{1}{\sqrt{1+t}}.$$

Do sales decline more slowly or more rapidly as time increases? Answer by referring to a sketch of the graph of $g(t)$ together with several tangent lines.

SOLUTION We notice from the figure below that, as time increases, the slopes of the tangent lines to the graph of $g(t)$ become less negative. Thus, sales decline more slowly as time increases.



24. The fraction of a city's population infected by a flu virus is plotted as a function of time (in weeks) in Figure 5.

(a) Which quantities are represented by the slopes of lines A and B? Estimate these slopes.

(b) Is the flu spreading more rapidly at $t = 1$, 2, or 3?

(c) Is the flu spreading more rapidly at $t = 4$, 5, or 6?

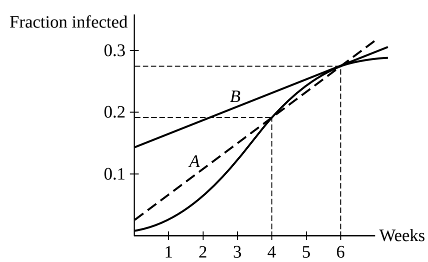


FIGURE 5

SOLUTION

(a) The slope of line A is the average rate of change over the interval $[4, 6]$, whereas the slope of the line B is the instantaneous rate of change at $t = 6$. Thus, the slope of the line $A \approx (0.28 - 0.19)/2 = 0.045$/week, whereas the slope of the line $B \approx (0.28 - 0.15)/6 = 0.0217$/week.

(b) Among times $t = 1, 2, 3$, the flu is spreading most rapidly at $t = 3$ since the slope is greatest at that instant; hence, the rate of change is greatest at that instant.

(c) Among times $t = 4, 5, 6$, the flu is spreading most rapidly at $t = 4$ since the slope is greatest at that instant; hence, the rate of change is greatest at that instant.

25. The graphs in Figure 6 represent the positions s of moving particles as functions of time t. Match each graph with a description:

(a) Speeding up

(b) Speeding up and then slowing down

(c) Slowing down

(d) Slowing down and then speeding up

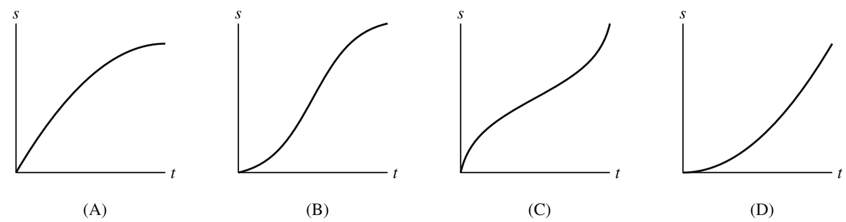


FIGURE 6

SOLUTION When a particle is speeding up over a time interval, its graph is bent upward over that interval. When a particle is slowing down, its graph is bent downward over that interval. Accordingly,

- In graph (A), the particle is (c) slowing down.
- In graph (B), the particle is (b) speeding up and then slowing down.
- In graph (C), the particle is (d) slowing down and then speeding up.
- In graph (D), the particle is (a) speeding up.

26. An epidemiologist finds that the percentage $N(t)$ of susceptible children who were infected on day t during the first three weeks of a measles outbreak is given, to a reasonable approximation, by the formula (Figure 7)

$$N(t) = \frac{100t^2}{t^3 + 5t^2 - 100t + 380}$$

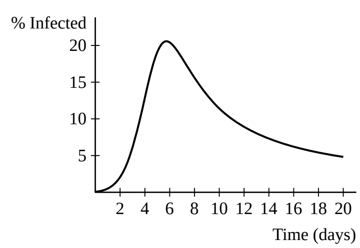


FIGURE 7 Graph of $N(t)$.

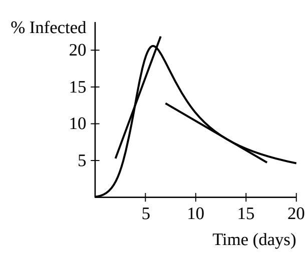
(a) Draw the secant line whose slope is the average rate of change in infected children over the intervals $[4, 6]$ and $[12, 14]$. Then compute these average rates (in units of percent per day).

(b) Is the rate of decline greater at $t = 8$ or $t = 16$?

(c) Estimate the rate of change of $N(t)$ on day 12.

SOLUTION

(a)



The average rate of change of $N(t)$ over the interval between day 4 and day 6 is given by

$$\frac{\Delta N}{\Delta t} = \frac{N(6) - N(4)}{6 - 4} = 3.776\%/\text{day}.$$

Similarly, we calculate the average rate of change of $N(t)$ over the interval between day 12 and day 14 as

$$\frac{\Delta N}{\Delta t} = \frac{N(14) - N(12)}{14 - 12} = -0.7983\%/\text{day}.$$

(b) The slope of the tangent line at $t = 8$ would be more negative than the slope of the tangent line at $t = 16$. Thus, the rate of decline is greater at $t = 8$ than at $t = 16$.

(c)

time interval	[12, 12.5]	[12, 12.2]	[12, 12.01]	[12, 12.001]
average rate of change	−0.9288	−0.9598	−0.9805	−0.9815

time interval	[11.5, 12]	[11.8, 12]	[11.99, 12]	[11.999, 12]
average rate of change	−1.0402	−1.0043	−0.9827	−0.9817

The instantaneous rate of change of $N(t)$ on day 12 is $-0.9816\%/\text{day}$.

27. The fungus *Fusarium exosporium* infects a field of flax plants through the roots and causes the plants to wilt. Eventually, the entire field is infected. The percentage $f(t)$ of infected plants as a function of time t (in days) since planting is shown in Figure 8.

(a) What are the units of the rate of change of $f(t)$ with respect to t? What does this rate measure?

(b) Use the graph to rank (from smallest to largest) the average infection rates over the intervals [0, 12], [20, 32], and [40, 52].

(c) Use the following table to compute the average rates of infection over the intervals [30, 40], [40, 50], [30, 50].

Days	0	10	20	30	40	50	60
Percent infected	0	18	56	82	91	96	98

(d) Draw the tangent line at $t = 40$ and estimate its slope.

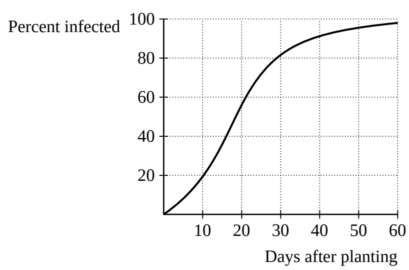


FIGURE 8

SOLUTION

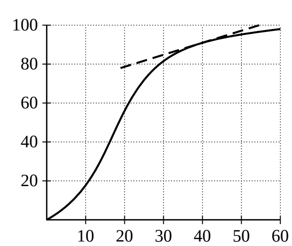
(a) The units of the rate of change of $f(t)$ with respect to t are percent /day or %/d. This rate measures how quickly the population of flax plants is becoming infected.

(b) From smallest to largest, the average rates of infection are those over the intervals [40, 52], [0, 12], [20, 32]. This is because the slopes of the secant lines over these intervals are arranged from smallest to largest.

(c) The average rates of infection over the intervals [30, 40], [40, 50], [30, 50] are 0.9, 0.5, 0.7 %/d, respectively.

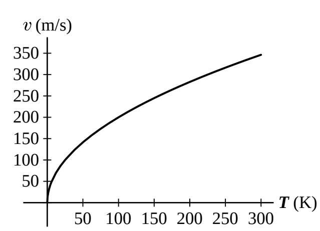
(d) The tangent line sketched in the graph below appears to pass through the points $(20, 80)$ and $(40, 91)$. The estimate of the instantaneous rate of infection at $t = 40$ days is therefore

$$\frac{91 - 80}{40 - 20} = \frac{11}{20} = 0.55\%/\text{d}.$$



28. Let $v = 20\sqrt{T}$ as in Example 2. Is the rate of change of v with respect to T greater at low temperatures or high temperatures? Explain in terms of the graph.

SOLUTION



As the graph progresses to the right, the graph bends progressively downward, meaning that the slope of the tangent lines becomes smaller. This means that the rate of change of v with respect to T is lower at high temperatures.

29. If an object in linear motion (but with changing velocity) covers Δs meters in Δt seconds, then its average velocity is $v_0 = \Delta s/\Delta t$ m/s. Show that it would cover the same distance if it traveled at constant velocity v_0 over the same time interval. This justifies our calling $\Delta s/\Delta t$ the *average velocity*.

SOLUTION At constant velocity, the distance traveled is equal to velocity times time, so an object moving at constant velocity v_0 for Δt seconds travels $v_0 \delta t$ meters. Since $v_0 = \Delta s/\Delta t$, we find

$$\text{distance traveled} = v_0 \delta t = \left(\frac{\Delta s}{\Delta t}\right) \Delta t = \Delta s$$

So the object covers the same distance Δs by traveling at constant velocity v_0.

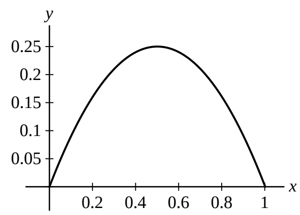
30. Sketch the graph of $f(x) = x(1-x)$ over $[0, 1]$. Refer to the graph and, without making any computations, find:

(a) The average rate of change over $[0, 1]$

(b) The (instantaneous) rate of change at $x = \frac{1}{2}$

(c) The values of x at which the rate of change is positive

SOLUTION



(a) $f(0) = f(1)$, so there is no change between $x = 0$ and $x = 1$. The average rate of change is zero.

(b) The tangent line to the graph of $f(x)$ is horizontal at $x = \frac{1}{2}$; the instantaneous rate of change is zero at this point.

(c) The rate of change is positive at all points where the graph is rising, because the slope of the tangent line is positive at these points. This is so for all x between $x = 0$ and $x = 0.5$.

31. Which graph in Figure 9 has the following property: For all x, the average rate of change over $[0, x]$ is greater than the instantaneous rate of change at x. Explain.

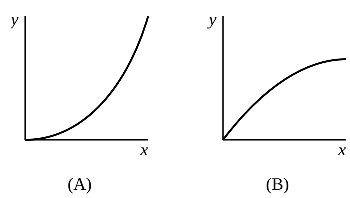


FIGURE 9

SOLUTION The average rate of change over $[0, x]$ is greater than the instantaneous rate of change at x: (B). The graph in (B) bends downward, so the slope of the secant line through $(0, 0)$ and $(x, f(x))$ is larger than the slope of the tangent line at $(x, f(x))$.

Further Insights and Challenges

32. The height of a projectile fired in the air vertically with initial velocity 25 m/s is

$$h(t) = 25t - 4.9t^2 \text{ m.}$$

(a) Compute $h(1)$. Show that $h(t) - h(1)$ can be factored with $(t - 1)$ as a factor.

(b) Using part (a), show that the average velocity over the interval $[1, t]$ is $20.1 - 4.9t$.

(c) Use this formula to find the average velocity over several intervals $[1, t]$ with t close to 1. Then estimate the instantaneous velocity at time $t = 1$.

SOLUTION

(a) With $h(t) = 25t - 4.9t^2$, we have $h(1) = 20.1$ m, so

$$h(t) - h(1) = -4.9t^2 + 25t - 20.1.$$

Factoring the quadratic, we obtain

$$h(t) - h(1) = (t-1)(-4.9t + 20.1).$$

(b) The average velocity over the interval $[1, t]$ is

$$\frac{h(t) - h(1)}{t-1} = \frac{(t-1)(-4.9t + 20.1)}{t-1} = 20.1 - 4.9t.$$

(c)

t	1.01	1.001	1.0001	1.00001
average velocity over $[1, t]$	15.151	15.1951	15.19951	15.199951

The instantaneous velocity is approximately 15.2 m/s. Plugging $t = 1$ second into the formula in (b) yields $20.1 - 4.9(1) = 15.2$ m/s exactly.

33. Let $Q(t) = t^2$. As in the previous exercise, find a formula for the average rate of change of Q over the interval $[1, t]$ and use it to estimate the instantaneous rate of change at $t = 1$. Repeat for the interval $[2, t]$ and estimate the rate of change at $t = 2$.

SOLUTION The average rate of change is

$$\frac{Q(t) - Q(1)}{t-1} = \frac{t^2 - 1}{t-1}.$$

Applying the difference of squares formula gives that the average rate of change is $((t+1)(t-1))/(t-1) = (t+1)$ for $t \neq 1$. As t gets closer to 1, this gets closer to $1 + 1 = 2$. The instantaneous rate of change is 2.

For $t_0 = 2$, the average rate of change is

$$\frac{Q(t) - Q(2)}{t-2} = \frac{t^2 - 4}{t-2},$$

which simplifies to $t + 2$ for $t \neq 2$. As t approaches 2, the average rate of change approaches 4. The instantaneous rate of change is therefore 4.

34. Show that the average rate of change of $f(x) = x^3$ over $[1, x]$ is equal to

$$x^2 + x + 1.$$

Use this to estimate the instantaneous rate of change of $f(x)$ at $x = 1$.

SOLUTION The average rate of change is

$$\frac{f(x) - f(1)}{x-1} = \frac{x^3 - 1}{x-1}.$$

Factoring the numerator as the difference of cubes means the average rate of change is

$$\frac{(x-1)(x^2 + x + 1)}{x-1} = x^2 + x + 1$$

(for all $x \neq 1$). The closer x gets to 1, the closer the average rate of change gets to $1^2 + 1 + 1 = 3$. The instantaneous rate of change is 3.

35. Find a formula for the average rate of change of $f(x) = x^3$ over $[2, x]$ and use it to estimate the instantaneous rate of change at $x = 2$.

SOLUTION The average rate of change is

$$\frac{f(x) - f(2)}{x-2} = \frac{x^3 - 8}{x-2}.$$

Applying the difference of cubes formula to the numerator, we find that the average rate of change is

$$\frac{(x^2 + 2x + 4)(x-2)}{x-2} = x^2 + 2x + 4$$

for $x \neq 2$. The closer x gets to 2, the closer the average rate of change gets to $2^2 + 2(2) + 4 = 12$.

36. Let $T = \frac{3}{2}\sqrt{L}$ as in Exercise 21. The numbers in the second column of the following table are increasing, and those in the last column are decreasing. Explain why in terms of the graph of T as a function of L. Also, explain graphically why the instantaneous rate of change at $L = 3$ lies between 0.4329 and 0.4331.

Average Rates of Change of T with Respect to L

Interval	Average rate of change	Interval	Average rate of change
[3, 3.2]	0.42603	[2.8, 3]	0.44048
[3, 3.1]	0.42946	[2.9, 3]	0.43668
[3, 3.001]	0.43298	[2.999, 3]	0.43305
[3, 3.0005]	0.43299	[2.9995, 3]	0.43303

SOLUTION Since the average rate of change is increasing on the intervals $[3, L]$ as L get close to 3, we know that the slopes of the secant lines between points on the graph over these intervals are increasing. The more rows we add with smaller intervals, the greater the average rate of change. This means that the instantaneous rate of change is probably greater than all of the numbers in this column.

Likewise, since the average rate of change is *decreasing* on the intervals $[L, 3]$ as L gets closer to 3, we know that the slopes of the secant lines between points over these intervals are decreasing. This means that the instantaneous rate of change is probably less than all the numbers in this column.

The tangent slope is somewhere between the greatest value in the first column and the least value in the second column. Hence, it is between 0.43299 and 0.43303. The first column underestimates the instantaneous rate of change by secant slopes; this estimate improves as L decreases toward $L = 3$. The second column overestimates the instantaneous rate of change by secant slopes; this estimate improves as L increases toward $L = 3$.

2.2 Limits: A Numerical and Graphical Approach

Preliminary Questions

1. What is the limit of $f(x) = 1$ as $x \to \pi$?

SOLUTION $\lim_{x\to\pi} 1 = 1$.

2. What is the limit of $g(t) = t$ as $t \to \pi$?

SOLUTION $\lim_{t\to\pi} t = \pi$.

3. Is $\lim\limits_{x\to 10} 20$ equal to 10 or 20?

SOLUTION $\lim_{x\to 10} 20 = 20$.

4. Can $f(x)$ approach a limit as $x \to c$ if $f(c)$ is undefined? If so, give an example.

SOLUTION Yes. The limit of a function f as $x \to c$ does not depend on what happens *at* $x = c$, only on the behavior of f as $x \to c$. As an example, consider the function

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

The function is clearly not defined at $x = 1$ but

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} (x + 1) = 2.$$

5. What does the following table suggest about $\lim\limits_{x\to 1-} f(x)$ and $\lim\limits_{x\to 1+} f(x)$?

x	0.9	0.99	0.999	1.1	1.01	1.001
$f(x)$	7	25	4317	3.0126	3.0047	3.00011

SOLUTION The values in the table suggest that $\lim_{x\to 1-} f(x) = \infty$ and $\lim_{x\to 1+} f(x) = 3$.

6. Can you tell whether $\lim\limits_{x\to 5} f(x)$ exists from a plot of $f(x)$ for $x > 5$? Explain.

SOLUTION No. By examining values of $f(x)$ for x close to but greater than 5, we can determine whether the one-sided limit $\lim_{x\to 5+} f(x)$ exists. To determine whether $\lim_{x\to 5} f(x)$ exists, we must examine value of $f(x)$ on both sides of $x = 5$.

7. If you know in advance that $\lim_{x\to 5} f(x)$ exists, can you determine its value from a plot of $f(x)$ for all $x > 5$?

SOLUTION Yes. If $\lim_{x\to 5} f(x)$ exists, then both one-sided limits must exist and be equal.

Exercises

In Exercises 1–4, fill in the tables and guess the value of the limit.

1. $\lim_{x\to 1} f(x)$, where $f(x) = \dfrac{x^3 - 1}{x^2 - 1}$.

x	$f(x)$	x	$f(x)$
1.002		0.998	
1.001		0.999	
1.0005		0.9995	
1.00001		0.99999	

SOLUTION

x	0.998	0.999	0.9995	0.99999	1.00001	1.0005	1.001	1.002
$f(x)$	1.498501	1.499250	1.499625	1.499993	1.500008	1.500375	1.500750	1.501500

The limit as $x \to 1$ is $\frac{3}{2}$.

2. $\lim_{t\to 0} h(t)$, where $h(t) = \dfrac{\cos t - 1}{t^2}$. Note that $h(t)$ is even; that is, $h(t) = h(-t)$.

t	± 0.002	± 0.0001	± 0.00005	± 0.00001
$h(t)$				

SOLUTION

t	± 0.002	± 0.0001
$h(t)$	−0.499999833333	−0.499999999583
t	± 0.00005	± 0.00001
$h(t)$	−0.499999999896	−0.500000000000

The limit as $t \to 0$ is $-\frac{1}{2}$.

3. $\lim_{y\to 2} f(y)$, where $f(y) = \dfrac{y^2 - y - 2}{y^2 + y - 6}$.

y	$f(y)$	y	$f(y)$
2.002		1.998	
2.001		1.999	
2.0001		1.9999	

SOLUTION

y	1.998	1.999	1.9999	2.0001	2.001	2.02
$f(y)$	0.59984	0.59992	0.599992	0.600008	0.60008	0.601594

The limit as $y \to 2$ is $\frac{3}{5}$.

4. $\lim_{x\to 0+} f(x)$, where $f(x) = x \ln x$.

x	1	0.5	0.1	0.05	0.01	0.005	0.001
$f(x)$							

SOLUTION

x	1.0	0.5	0.1	0.05	0.01	0.005	0.001
$f(x)$	0	−0.34657	−0.23026	−0.14979	−0.04605	−0.02649	−0.00691

The limit as $x \to 0+$ is 0.

5. Determine $\lim_{x \to 0.5} f(x)$ for $f(x)$ as in Figure 1.

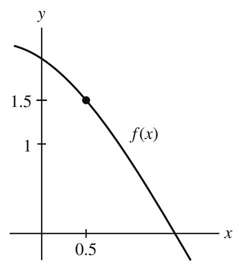


FIGURE 1

SOLUTION The graph suggests that $f(x) \to 1.5$ as $x \to 0.5$.

6. Determine $\lim_{x \to 0.5} g(x)$ for $g(x)$ as in Figure 2.

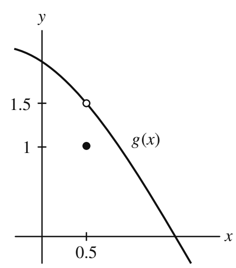


FIGURE 2

SOLUTION The graph suggests that $g(x) \to 1.5$ as $x \to 0.5$. The value $g(0.5)$, which happens to be 1, does not affect the limit.

In Exercises 7 and 8, evaluate the limit.

7. $\lim_{x \to 21} x$

SOLUTION As $x \to 21$, $f(x) = x \to 21$. You can see this, for example, on the graph of $f(x) = x$.

8. $\lim_{x \to 4.2} \sqrt{3}$

SOLUTION The graph of $f(x) = \sqrt{3}$ is a horizontal line. $f(x) = \sqrt{3}$ for all values of x, so the limit is also equal to $\sqrt{3}$.

In Exercises 9–16, verify each limit using the limit definition. For example, in Exercise 9, show that $|3x - 12|$ *can be made as small as desired by taking* x *close to* 4.

9. $\lim_{x \to 4} 3x = 12$

SOLUTION $|3x - 12| = 3|x - 4|$. $|3x - 12|$ can be made arbitrarily small by making x close enough to 4, thus making $|x - 4|$ small.

10. $\lim_{x \to 5} 3 = 3$

SOLUTION $|f(x) - 3| = |3 - 3| = 0$ for all values of x so $f(x) - 3$ is already smaller than any positive number as $x \to 5$.

11. $\lim_{x \to 3} (5x + 2) = 17$

SOLUTION $|(5x + 2) - 17| = |5x - 15| = 5|x - 3|$. Therefore, if you make $|x - 3|$ small enough, you can make $|(5x + 2) - 17|$ as small as desired.

12. $\lim_{x \to 2} (7x - 4) = 10$

SOLUTION As $x \to 2$, note that $|(7x-4)-10| = |7x-14| = 7\,|x-2|$. If you make $|x-2|$ small enough, you can make $|(7x-4)-10|$ as small as desired.

13. $\lim_{x \to 0} x^2 = 0$

SOLUTION As $x \to 0$, we have $|x^2 - 0| = |x+0||x-0|$. To simplify things, suppose that $|x| < 1$, so that $|x+0||x-0| = |x||x| < |x|$. By making $|x|$ sufficiently small, so that $|x+0||x-0| = x^2$ is even smaller, you can make $|x^2 - 0|$ as small as desired.

14. $\lim_{x \to 0} (3x^2 - 9) = -9$

SOLUTION $|3x^2 - 9 - (-9)| = |3x^2| = 3|x^2|$. If you make $|x| < 1$, $|x^2| < |x|$, so that making $|x-0|$ small enough can make $|3x^2 - 9 - (-9)|$ as small as desired.

15. $\lim_{x \to 0} (4x^2 + 2x + 5) = 5$

SOLUTION As $x \to 0$, we have $|4x^2 + 2x + 5 - 5| = |4x^2 + 2x| = |x||4x+2|$. If $|x| < 1$, $|4x+2|$ can be no bigger than 6, so $|x||4x+2| < 6|x|$. Therefore, by making $|x-0| = |x|$ sufficiently small, you can make $|4x^2 + 2x + 5 - 5| = |x||4x+2|$ as small as desired.

16. $\lim_{x \to 0} (x^3 + 12) = 12$

SOLUTION $|(x^3 + 12) - 12| = |x^3|$. If we make $|x| < 1$, then $|x^3| < |x|$. Therefore, by making $|x-0| = |x|$ sufficiently small, we can make $|(x^3+12) - 12|$ as small as desired.

In Exercises 17–36, estimate the limit numerically or state that the limit does not exist. If infinite, state whether the one-sided limits are ∞ or $-\infty$.

17. $\lim_{x \to 1} \dfrac{\sqrt{x} - 1}{x - 1}$

SOLUTION

x	0.9995	0.99999	1.00001	1.0005
$f(x)$	0.500063	0.500001	0.49999	0.499938

The limit as $x \to 1$ is $\frac{1}{2}$.

18. $\lim_{x \to -4} \dfrac{2x^2 - 32}{x + 4}$

SOLUTION

x	−4.001	−4.0001	−3.9999	−3.999
$f(x)$	−16.002	−16.0002	−15.9998	−15.998

The limit as $x \to -4$ is -16.

19. $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x^2 - x - 2}$

SOLUTION

x	1.999	1.99999	2.00001	2.001
$f(x)$	1.666889	1.666669	1.666664	1.666445

The limit as $x \to 2$ is $\frac{5}{3}$.

20. $\lim_{x \to 3} \dfrac{x^3 - 2x^2 - 9}{x^2 - 2x - 3}$

SOLUTION

x	2.99	2.995	3.005	3.01
$f(x)$	3.741880	3.745939	3.754064	3.758130

The limit as $x \to 3$ is 3.75.

21. $\lim_{x\to 0} \frac{\sin 2x}{x}$

SOLUTION

x	-0.01	-0.005	0.005	0.01
$f(x)$	1.999867	1.999967	1.999967	1.999867

The limit as $x \to 0$ is 2.

22. $\lim_{x\to 0} \frac{\sin 5x}{x}$

SOLUTION

x	-0.01	-0.005	0.005	0.01
$f(x)$	4.997917	4.999479	4.999479	4.997917

The limit as $x \to 0$ is 5.

23. $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta}$

SOLUTION

θ	-0.05	-0.001	0.001	0.05
$f(\theta)$	0.0249948	0.0005	-0.0005	-0.0249948

The limit as $\theta \to 0$ is 0.

24. $\lim_{x\to 0} \frac{\sin x}{x^2}$

SOLUTION

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(x)$	-99.9983	-999.9998	-10000.0	10000.0	999.9998	99.9983

The limit does not exist. As $x \to 0-$, $f(x) \to -\infty$; similarly, as $x \to 0+$, $f(x) \to \infty$.

25. $\lim_{x\to 4} \frac{1}{(x-4)^3}$

SOLUTION

x	3.99	3.999	3.9999	4.0001	4.001	4.01
$f(x)$	-10^6	-10^9	-10^{12}	10^{12}	10^9	10^6

The limit does not exist. As $x \to 4-$, $f(x) \to -\infty$; similarly, as $x \to 4+$, $f(x) \to \infty$.

26. $\lim_{x\to 1-} \frac{3-x}{x-1}$

SOLUTION

x	0.99	0.999	0.9999	0.99999
$f(x)$	-201	-2001	-20001	-200001

As $x \to 1-$, $f(x) \to -\infty$.

27. $\lim_{x\to 3+} \frac{x-4}{x^2-9}$

SOLUTION

x	3.01	3.001	3.0001	3.00001
$f(x)$	-16.473	-166.473	-1666.473	-16666.473

As $x \to 3+$, $f(x) \to -\infty$.

28. $\lim_{h\to 0} \dfrac{3^h - 1}{h}$

SOLUTION

h	−0.05	−0.001	−0.0001	0.0001	0.001	0.05
$f(h)$	1.06898	1.09801	1.09855	1.09867	1.09922	1.12935

The limit as $h \to 0$ is approximately 1.099. (The exact answer is $\ln 3$.)

29. $\lim_{h\to 0} \sin h \cos \dfrac{1}{h}$

SOLUTION

h	−0.01	−0.001	−0.0001	0.0001	0.001	0.01
$f(h)$	−0.008623	−0.000562	0.000095	−0.000095	0.000562	0.008623

The limit as $h \to 0$ is 0.

30. $\lim_{h\to 0} \cos \dfrac{1}{h}$

SOLUTION

h	±0.1	±0.01	±0.001	±0.0001
$f(h)$	−0.839072	0.862319	0.562379	−0.952155

The limit does not exist since $\cos(1/h)$ oscillates infinitely often as $h \to 0$.

31. $\lim_{x\to 0} |x|^x$

SOLUTION

x	−0.05	−0.001	−0.00001	0.00001	0.001	0.05
$f(x)$	1.161586	1.006932	1.000115	0.999885	0.993116	0.860892

The limit as $x \to 0$ is 1.

32. $\lim_{x\to 1+} \dfrac{\sec^{-1} x}{\sqrt{x-1}}$

SOLUTION

x	1.05	1.01	1.005	1.001
$f(x)$	1.3857	1.4084	1.4113	1.4136

The limit as $x \to 1+$ is approximately 1.414. (The exact answer is $\sqrt{2}$.)

33. $\lim_{t\to e} \dfrac{t - e}{\ln t - 1}$

SOLUTION

r	$e - 0.01$	$e - 0.001$	$e - 0.0001$	$e + 0.0001$	$e + 0.001$	$e + 0.01$
$f(t)$	2.713279	2.717782	2.718232	2.718332	2.718782	2.723279

The limit as $t \to e$ is approximately 2.718. (The exact answer is e.)

34. $\lim_{r\to 0} (1 + r)^{1/r}$

SOLUTION

r	−0.01	−0.001	−0.0001	0.0001	0.001	0.01
$f(r)$	2.731999	2.719642	2.718418	2.718146	2.716924	2.704814

The limit as $r \to 0$ is approximately 2.718. (The exact answer is e.)

35. $\lim_{x\to 1-} \dfrac{\tan^{-1} x}{\cos^{-1} x}$

SOLUTION

x	0.999	0.9999	0.99999	0.999999	0.9999999
$f(x)$	17.549	55.532	175.619	555.360	1756.204

The limit as $x \to 1-$ does not exist.

36. $\lim_{x\to 0} \dfrac{\tan^{-1} x - x}{\sin^{-1} x - x}$

SOLUTION

x	−0.01	−0.001	0.001	0.01
$f(x)$	−1.999791	−2.000066	−2.000066	−1.999791

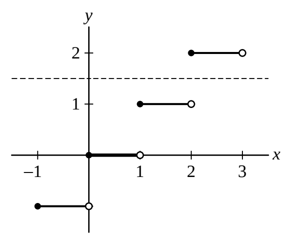
The limit as $x \to 0$ is approximately -2.00. (The exact answer is -2.)

37. The **greatest integer function** is defined by $[x] = n$, where n is the unique integer such that $n \le x < n + 1$. Sketch the graph of $y = [x]$. Calculate, for c an integer:

(a) $\lim_{x\to c-} [x]$

(b) $\lim_{x\to c+} [x]$

SOLUTION Here is a graph of the greatest integer function:



(a) From the graph, we see that, for c an integer,

$$\lim_{x\to c-} [x] = c - 1.$$

(b) From the graph, we see that, for c an integer,

$$\lim_{x\to c+} [x] = c.$$

38. Determine the one-sided limits at $c = 1$, 2, and 4 of the function $g(x)$ shown in Figure 3, and state whether the limit exists at these points.

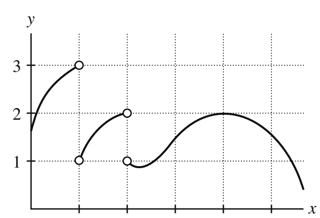


FIGURE 3

SOLUTION

- At $c = 1$, the left-hand limit is $\lim_{x\to 1-} g(x) = 3$, whereas the right-hand limit is $\lim_{x\to 1+} g(x) = 1$. Accordingly, the two-sided limit does not exist at $c = 1$.
- At $c = 2$, the left-hand limit is $\lim_{x\to 2-} g(x) = 2$, whereas the right-hand limit is $\lim_{x\to 2+} g(x) = 1$. Accordingly, the two-sided limit does not exist at $c = 2$.
- At $c = 4$, the left-hand limit is $\lim_{x\to 4-} g(x) = 2$, whereas the right-hand limit is $\lim_{x\to 4+} g(x) = 2$. Accordingly, the two-sided limit exists at $c = 4$ and equals 2.

In Exercises 39–46, determine the one-sided limits numerically or graphically. If infinite, state whether the one-sided limits are ∞ or $-\infty$, and describe the corresponding vertical asymptote. In Exercise 46, $[x]$ is the greatest integer function defined in Exercise 37.

39. $\lim_{x\to 0\pm} \frac{\sin x}{|x|}$

SOLUTION

x	−0.2	−0.02	0.02	0.2
$f(x)$	−0.993347	−0.999933	0.999933	0.993347

The left-hand limit is $\lim_{x\to 0-} f(x) = -1$, whereas the right-hand limit is $\lim_{x\to 0+} f(x) = 1$.

40. $\lim_{x\to 0\pm} |x|^{1/x}$

SOLUTION

x	−0.2	−0.1	0.15	0.2
$f(x)$	3125.0	10^{10}	0.000003	0.000320

The left-hand limit is $\lim_{x\to 0-} f(x) = \infty$, whereas the right-hand limit is $\lim_{x\to 0+} f(x) = 0$. Thus, the line $x = 0$ is a vertical asymptote from the left for the graph of $y = |x|^{1/x}$.

41. $\lim_{x\to 0\pm} \frac{x - \sin|x|}{x^3}$

SOLUTION

x	−0.1	−0.01	0.01	0.1
$f(x)$	199.853	19999.8	0.166666	0.166583

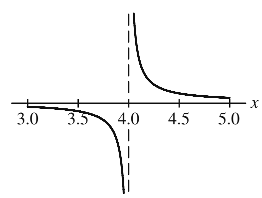
The left-hand limit is $\lim_{x\to 0-} f(x) = \infty$, whereas the right-hand limit is $\lim_{x\to 0+} f(x) = \frac{1}{6}$. Thus, the line $x = 0$ is a vertical asymptote from the left for the graph of $y = \frac{x-\sin|x|}{x^3}$.

42. $\lim_{x\to 4\pm} \frac{x+1}{x-4}$

SOLUTION The graph of $y = \frac{x+1}{x-4}$ for x near 4 is shown below. From this graph, we see that

$$\lim_{x\to 4-} \frac{x+1}{x-4} = -\infty \quad \text{while} \quad \lim_{x\to 4+} \frac{x+1}{x-4} = \infty.$$

Thus, the line $x = 4$ is a vertical asymptote for the graph of $y = \frac{x+1}{x-4}$.

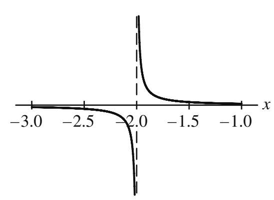


43. $\lim_{x\to -2\pm} \frac{4x^2+7}{x^3+8}$

SOLUTION The graph of $y = \frac{4x^2+7}{x^3+8}$ for x near −2 is shown below. From this graph, we see that

$$\lim_{x\to -2-} \frac{4x^2+7}{x^3+8} = -\infty \quad \text{while} \quad \lim_{x\to -2+} \frac{4x^2+7}{x^3+8} = \infty.$$

Thus, the line $x = -2$ is a vertical asymptote for the graph of $y = \frac{4x^2+7}{x^3+8}$.

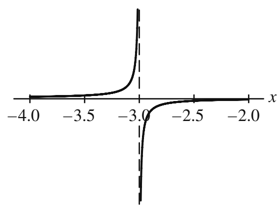


44. $\lim_{x\to -3\pm} \frac{x^2}{x^2-9}$

SOLUTION The graph of $y = \frac{x^2}{x^2-9}$ for x near -3 is shown below. From this graph, we see that

$$\lim_{x\to -3-} \frac{x^2}{x^2-9} = \infty \quad \text{while} \quad \lim_{x\to -3+} \frac{x^2}{x^2-9} = -\infty.$$

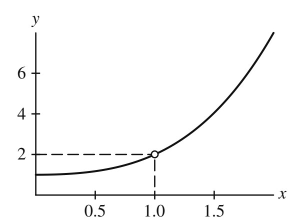
Thus, the line $x = -3$ is a vertical asymptote for the graph of $y = \frac{x^2}{x^2-9}$.



45. $\lim_{x\to 1\pm} \frac{x^5+x-2}{x^2+x-2}$

SOLUTION The graph of $y = \frac{x^5+x-2}{x^2+x-2}$ for x near 1 is shown below. From this graph, we see that

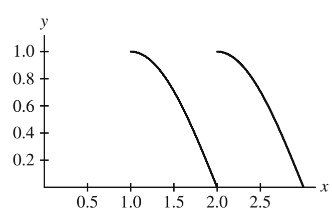
$$\lim_{x\to 1\pm} \frac{x^5+x-2}{x^2+x-2} = 2.$$



46. $\lim_{x\to 2\pm} \cos\left(\frac{\pi}{2}(x-[x])\right)$

SOLUTION The graph of $y = \cos\left(\frac{\pi}{2}(x-[x])\right)$ for x near 2 is shown below. From this graph, we see that

$$\lim_{x\to 2-} \cos\left(\frac{\pi}{2}(x-[x])\right) = 0 \quad \text{while} \quad \lim_{x\to 2+} \cos\left(\frac{\pi}{2}(x-[x])\right) = 1.$$



47. Determine the one-sided limits at $c = 2, 4$ of the function $f(x)$ in Figure 4. What are the vertical asymptotes of $f(x)$?

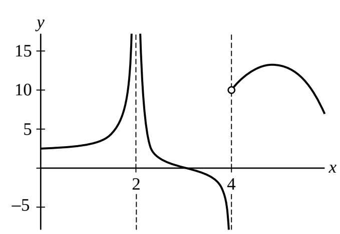


FIGURE 4

SOLUTION

- For $c = 2$, we have $\lim_{x\to 2-} f(x) = \infty$ and $\lim_{x\to 2+} f(x) = \infty$.
- For $c = 4$, we have $\lim_{x\to 4-} f(x) = -\infty$ and $\lim_{x\to 4+} f(x) = 10$.

The vertical asymptotes are the vertical lines $x = 2$ and $x = 4$.

48. Determine the infinite one- and two-sided limits in Figure 5.

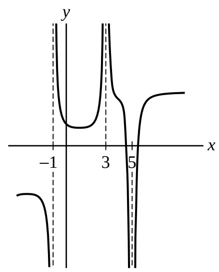


FIGURE 5

SOLUTION

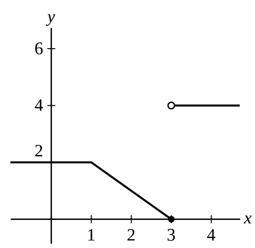
- $\lim_{x\to -1-} f(x) = -\infty$
- $\lim_{x\to -1+} f(x) = \infty$
- $\lim_{x\to 3} f(x) = \infty$
- $\lim_{x\to 5} f(x) = -\infty$

The vertical asymptotes are the vertical lines $x = 1$, $x = 3$, and $x = 5$.

In Exercises 49–52, sketch the graph of a function with the given limits.

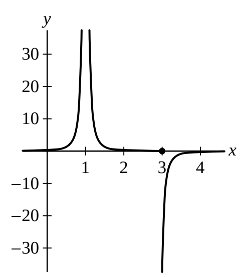
49. $\lim_{x\to 1} f(x) = 2, \quad \lim_{x\to 3-} f(x) = 0, \quad \lim_{x\to 3+} f(x) = 4$

SOLUTION



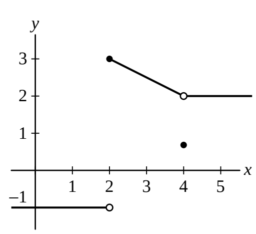
50. $\lim_{x\to 1} f(x) = \infty, \quad \lim_{x\to 3-} f(x) = 0, \quad \lim_{x\to 3+} f(x) = -\infty$

SOLUTION



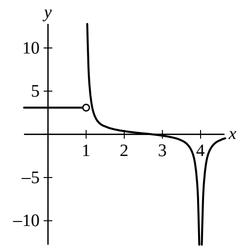
51. $\lim_{x\to 2+} f(x) = f(2) = 3, \quad \lim_{x\to 2-} f(x) = -1, \lim_{x\to 4} f(x) = 2 \neq f(4)$

SOLUTION



52. $\lim_{x\to 1+} f(x) = \infty, \quad \lim_{x\to 1-} f(x) = 3, \quad \lim_{x\to 4} f(x) = -\infty$

SOLUTION



53. Determine the one-sided limits of the function $f(x)$ in Figure 6, at the points $c = 1, 3, 5, 6$.

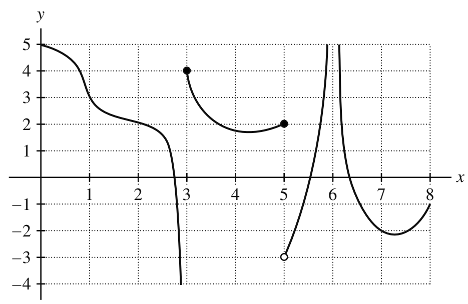


FIGURE 6 Graph of $f(x)$

SOLUTION

- $\lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) = 3$
- $\lim_{x\to 3-} f(x) = -\infty$
- $\lim_{x\to 3+} f(x) = 4$
- $\lim_{x\to 5-} f(x) = 2$
- $\lim_{x\to 5+} f(x) = -3$
- $\lim_{x\to 6-} f(x) = \lim_{x\to 6+} f(x) = \infty$

54. Does either of the two oscillating functions in Figure 7 appear to approach a limit as $x \to 0$?

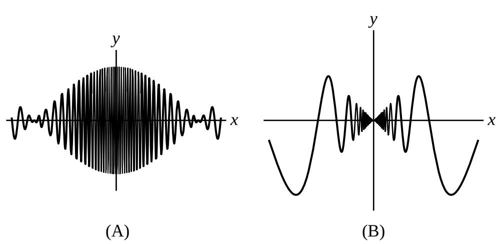


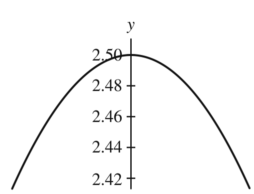
FIGURE 7

SOLUTION (A) does not appear to approach a limit as $x \to 0$; the values of the function oscillate wildly as $x \to 0$. The values of the function graphed in (B) seem to settle to 0 as $x \to 0$, so the limit seems to exist.

GU *In Exercises 55–60, plot the function and use the graph to estimate the value of the limit.*

55. $\lim_{\theta\to 0} \dfrac{\sin 5\theta}{\sin 2\theta}$

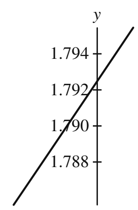
SOLUTION



From the graph of $y = \dfrac{\sin 5\theta}{\sin 2\theta}$ shown above, we see that the limit as $\theta \to 0$ is $\frac{5}{2}$.

56. $\displaystyle\lim_{x\to 0} \frac{12^x - 1}{4^x - 1}$

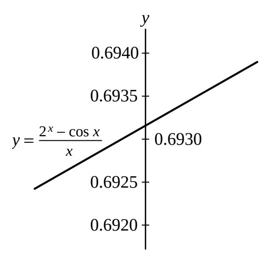
SOLUTION



From the graph of $y = \dfrac{12^x - 1}{4^x - 1}$ shown above, we see that the limit as $x \to 0$ is approximately 1.7925. (The exact answer is $\ln 12/\ln 4$.)

57. $\displaystyle\lim_{x\to 0} \frac{2^x - \cos x}{x}$

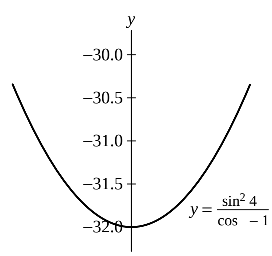
SOLUTION



The limit as $x \to 0$ is approximately 0.693. (The exact answer is $\ln 2$.)

58. $\displaystyle\lim_{\theta\to 0} \frac{\sin^2 4\theta}{\cos\theta - 1}$

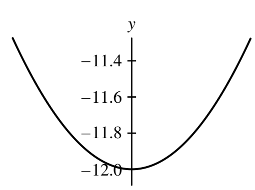
SOLUTION



The limit as $\theta \to 0$ is -32.

59. $\displaystyle\lim_{\theta\to 0} \frac{\cos 7\theta - \cos 5\theta}{\theta^2}$

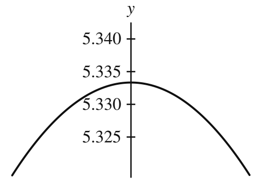
SOLUTION



From the graph of $y = \dfrac{\cos 7\theta - \cos 5\theta}{\theta^2}$ shown above, we see that the limit as $\theta \to 0$ is -12.

60. $\displaystyle\lim_{\theta\to 0} \frac{\sin^2 2\theta - \theta \sin 4\theta}{\theta^4}$

SOLUTION



From the graph of $y = \dfrac{\sin^2 2\theta - \theta \sin 4\theta}{\theta^4}$ shown above, we see that the limit as $\theta \to 0$ is approximately 5.333. (The exact answer is $\frac{16}{3}$.)

61. Let n be a positive integer. For which n are the two infinite one-sided limits $\lim_{x\to 0\pm} 1/x^n$ equal?

SOLUTION First, suppose that n is even. Then $x^n \geq 0$ for all x, and $\frac{1}{x^n} > 0$ for all $x \neq 0$. Hence,

$$\lim_{x\to 0-} \frac{1}{x^n} = \lim_{x\to 0+} \frac{1}{x^n} = \infty.$$

Next, suppose that n is odd. Then $\frac{1}{x^n} > 0$ for all $x > 0$ but $\frac{1}{x^n} < 0$ for all $x < 0$. Thus,

$$\lim_{x\to 0-} \frac{1}{x^n} = -\infty \quad \text{but} \quad \lim_{x\to 0+} \frac{1}{x^n} = \infty.$$

Finally, the two infinite one-sided limits are equal whenever n is even.

62. Let $L(n) = \lim_{x\to 1} \left(\dfrac{n}{1-x^n} - \dfrac{1}{1-x} \right)$ for n a positive integer. Investigate $L(n)$ numerically for several values of n, and then guess the value of of $L(n)$ in general.

SOLUTION

- We first notice that for $n = 1$,

$$\frac{1}{1-x} - \frac{1}{1-x} = 0,$$

so $L(1) = 0$.

- Next, let's try $n = 3$. From the table below, it appears that $L(3) = 1$.

x	0.99	0.999	1.001	1.01
$f(x)$	1.006700	1.000667	0.999334	0.993367

- For $n = 6$, we find

x	0.99	0.999	0.9999	1.0001	1.001	1.01
$f(x)$	2.529312	2.502919	2.500392	2.499375	2.497082	2.470980

Thus, $L(6) = 2.5 = \frac{5}{2}$

From these values, we conjecture that $L(n) = \frac{n-1}{2}$.

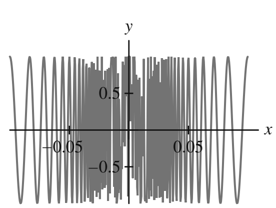
63. GU In some cases, numerical investigations can be misleading. Plot $f(x) = \cos \frac{\pi}{x}$.

(a) Does $\lim_{x\to 0} f(x)$ exist?

(b) Show, by evaluating $f(x)$ at $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$, that you might be able to trick your friends into believing that the limit exists and is equal to $L = 1$.

(c) Which sequence of evaluations might trick them into believing that the limit is $L = -1$.

SOLUTION Here is the graph of $f(x)$.



(a) From the graph of $f(x)$, which shows that the value of $f(x)$ oscillates more and more rapidly as $x \to 0$, it follows that $\lim_{x\to 0} f(x)$ does not exist.

(b) Notice that

$$f\left(\pm\frac{1}{2}\right) = \cos \pm\frac{\pi}{1/2} = \cos \pm 2\pi = 1;$$

$$f\left(\pm\frac{1}{4}\right) = \cos \pm\frac{\pi}{1/4} = \cos \pm 4\pi = 1;$$

$$f\left(\pm\frac{1}{6}\right) = \cos \pm\frac{\pi}{1/6} = \cos \pm 6\pi = 1;$$

and, in general, $f(\pm\frac{1}{2n}) = 1$ for all integers n.

(c) At $x = \pm 1, \pm\frac{1}{3}, \pm\frac{1}{5}, \ldots$, the value of $f(x)$ is always -1.

Further Insights and Challenges

64. Light waves of frequency λ passing through a slit of width a produce a **Fraunhofer diffraction pattern** of light and dark fringes (Figure 8). The intensity as a function of the angle θ is

$$I(\theta) = I_m\left(\frac{\sin(R\sin\theta)}{R\sin\theta}\right)^2$$

where $R = \pi a/\lambda$ and I_m is a constant. Show that the intensity function is not defined at $\theta = 0$. Then choose any two values for R and check numerically that $I(\theta)$ approaches I_m as $\theta \to 0$.

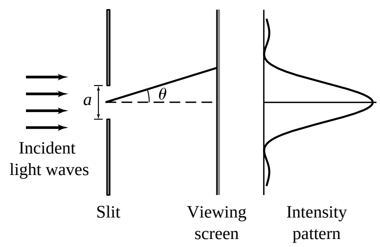


FIGURE 8 Fraunhofer diffraction pattern.

SOLUTION If you plug in $\theta = 0$, you get a division by zero in the expression

$$\frac{\sin\left(R\sin\theta\right)}{R\sin\theta};$$

thus, $I(0)$ is undefined. If $R = 2$, a table of values as $\theta \to 0$ follows:

θ	-0.01	-0.005	0.005	0.01
$I(\theta)$	$0.998667\ I_m$	$0.9999667\ I_m$	$0.9999667\ I_m$	$0.9998667\ I_m$

The limit as $\theta \to 0$ is $1 \cdot I_m = I_m$.

If $R = 3$, the table becomes:

θ	-0.01	-0.005	0.005	0.01
$I(\theta)$	$0.999700\ I_m$	$0.999925\ I_m$	$0.999925\ I_m$	$0.999700\ I_m$

Again, the limit as $\theta \to 0$ is $1I_m = I_m$.

65. Investigate $\lim_{\theta\to 0}\frac{\sin n\theta}{\theta}$ numerically for several values of n. Then guess the value in general.

SOLUTION

- For $n = 3$, we have

θ	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\frac{\sin n\theta}{\theta}$	2.955202	2.999550	2.999996	2.999996	2.999550	2.955202

The limit as $\theta \to 0$ is 3.

- For $n = -5$, we have

θ	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\dfrac{\sin n\theta}{\theta}$	-4.794255	-4.997917	-4.999979	-4.999979	-4.997917	-4.794255

The limit as $\theta \to 0$ is -5.

- We surmise that, in general, $\lim_{\theta\to 0} \dfrac{\sin n\theta}{\theta} = n$.

66. Show numerically that $\lim_{x\to 0} \dfrac{b^x - 1}{x}$ for $b = 3, 5$ appears to equal $\ln 3$, $\ln 5$, where $\ln x$ is the natural logarithm. Then make a conjecture (guess) for the value in general and test your conjecture for two additional values of b.

SOLUTION

-

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\dfrac{5^x - 1}{x}$	1.486601	1.596556	1.608144	1.610734	1.622459	1.746189

We have $\ln 5 \approx 1.6094$.

-

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\dfrac{3^x - 1}{x}$	1.040415	1.092600	1.098009	1.099216	1.104669	1.161232

We have $\ln 3 \approx 1.0986$.

- We conjecture that $\lim_{x\to 0} \dfrac{b^x - 1}{x} = \ln b$ for any positive number b. Here are two additional test cases.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\dfrac{\left(\frac{1}{2}\right)^x - 1}{x}$	-0.717735	-0.695555	-0.693387	-0.692907	-0.690750	-0.669670

We have $\ln \frac{1}{2} \approx -0.69315$.

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\dfrac{7^x - 1}{x}$	1.768287	1.927100	1.944018	1.947805	1.964966	2.148140

We have $\ln 7 \approx 1.9459$.

67. Investigate $\lim_{x\to 1} \dfrac{x^n - 1}{x^m - 1}$ for (m, n) equal to $(2, 1)$, $(1, 2)$, $(2, 3)$, and $(3, 2)$. Then guess the value of the limit in general and check your guess for two additional pairs.

SOLUTION

-

x	0.99	0.9999	1.0001	1.01
$\dfrac{x - 1}{x^2 - 1}$	0.502513	0.500025	0.499975	0.497512

The limit as $x \to 1$ is $\frac{1}{2}$.

x	0.99	0.9999	1.0001	1.01
$\dfrac{x^2 - 1}{x - 1}$	1.99	1.9999	2.0001	2.01

The limit as $x \to 1$ is 2.

x	0.99	0.9999	1.0001	1.01
$\frac{x^2-1}{x^3-1}$	0.670011	0.666700	0.666633	0.663344

The limit as $x \to 1$ is $\frac{2}{3}$.

x	0.99	0.9999	1.0001	1.01
$\frac{x^3-1}{x^2-1}$	1.492513	1.499925	1.500075	1.507512

The limit as $x \to 1$ is $\frac{3}{2}$.

- For general m and n, we have $\lim_{x\to 1} \frac{x^n-1}{x^m-1} = \frac{n}{m}$.
-

x	0.99	0.9999	1.0001	1.01
$\frac{x-1}{x^3-1}$	0.336689	0.333367	0.333300	0.330022

The limit as $x \to 1$ is $\frac{1}{3}$.

x	0.99	0.9999	1.0001	1.01
$\frac{x^3-1}{x-1}$	2.9701	2.9997	3.0003	3.0301

The limit as $x \to 1$ is 3.

x	0.99	0.9999	1.0001	1.01
$\frac{x^3-1}{x^7-1}$	0.437200	0.428657	0.428486	0.420058

The limit as $x \to 1$ is $\frac{3}{7} \approx 0.428571$.

68. Find by numerical experimentation the positive integers k such that $\lim_{x\to 0} \frac{\sin(\sin^2 x)}{x^k}$ exists.

SOLUTION

- For $k = 1$, we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(\sin^2 x)}{x} = 0$.

x	−0.01	−0.0001	0.0001	0.01
$f(x)$	−0.01	−0.0001	0.0001	0.01

- For $k = 2$, we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(\sin^2 x)}{x^2} = 1$.

x	−0.01	−0.0001	0.0001	0.01
$f(x)$	0.999967	1.000000	1.000000	0.999967

- For $k = 3$, the limit does not exist.

x	−0.01	−0.0001	0.0001	0.01
$f(x)$	-10^2	-10^4	10^4	10^2

Indeed, as $x \to 0-$, $f(x) = \frac{\sin(\sin^2 x)}{x^3} \to -\infty$, whereas as $x \to 0+$, $f(x) = \frac{\sin(\sin^2 x)}{x^3} \to \infty$.

- For $k = 4$, we have $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin(\sin^2 x)}{x^4} = \infty$.

x	-0.01	-0.0001	0.0001	0.01
$f(x)$	10^4	10^8	10^8	10^4

- For $k = 5$, the limit does not exist.

x	-0.01	-0.0001	0.0001	0.01
$f(x)$	-10^6	-10^{12}	10^{12}	10^6

Indeed, as $x \to 0-$, $f(x) = \frac{\sin(\sin^2 x)}{x^5} \to -\infty$, whereas as $x \to 0+$, $f(x) = \frac{\sin(\sin^2 x)}{x^5} \to \infty$.

- For $k = 6$, we have $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin(\sin^2 x)}{x^6} = \infty$.

x	-0.01	-0.0001	0.0001	0.01
$f(x)$	10^8	10^{16}	10^{16}	10^8

- SUMMARY
  - For $k = 1$, the limit is 0.
  - For $k = 2$, the limit is 1.
  - For odd $k > 2$, the limit does not exist.
  - For even $k > 2$, the limit is ∞.

69. GU Plot the graph of $f(x) = \frac{2^x - 8}{x - 3}$.

(a) Zoom in on the graph to estimate $L = \lim_{x \to 3} f(x)$.

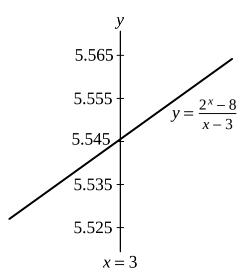
(b) Explain why

$$f(2.99999) \leq L \leq f(3.00001)$$

Use this to determine L to three decimal places.

SOLUTION

(a)



(b) It is clear that the graph of f rises as we move to the right. Mathematically, we may express this observation as: whenever $u < v$, $f(u) < f(v)$. Because

$$2.99999 < 3 = \lim_{x \to 3} f(x) < 3.00001,$$

it follows that

$$f(2.99999) < L = \lim_{x \to 3} f(x) < f(3.00001).$$

With $f(2.99999) \approx 5.54516$ and $f(3.00001) \approx 5.545195$, the above inequality becomes $5.54516 < L < 5.545195$; hence, to three decimal places, $L = 5.545$.

70. GU The function $f(x) = \frac{2^{1/x} - 2^{-1/x}}{2^{1/x} + 2^{-1/x}}$ is defined for $x \neq 0$.

(a) Investigate $\lim_{x \to 0+} f(x)$ and $\lim_{x \to 0-} f(x)$ numerically.

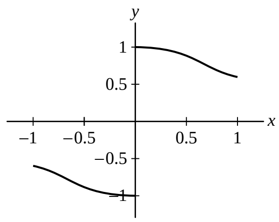
(b) Plot the graph of f and describe its behavior near $x = 0$.

SOLUTION

(a)

x	-0.3	-0.2	-0.1	0.1	0.2	0.3
$f(x)$	-0.980506	-0.998049	-0.999998	0.999998	0.998049	0.980506

(b) As $x \to 0-$, $f(x) \to -1$, whereas as $x \to 0+$, $f(x) \to 1$.



2.3 Basic Limit Laws

Preliminary Questions

1. State the Sum Law and Quotient Law.

SOLUTION Suppose $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist. The Sum Law states that

$$\lim_{x\to c} (f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

Provided $\lim_{x\to c} g(x) \neq 0$, the Quotient Law states that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

2. Which of the following is a verbal version of the Product Law (assuming the limits exist)?

(a) The product of two functions has a limit.

(b) The limit of the product is the product of the limits.

(c) The product of a limit is a product of functions.

(d) A limit produces a product of functions.

SOLUTION The verbal version of the Product Law is **(b)**: The limit of the product is the product of the limits.

3. Which statement is correct? The Quotient Law does not hold if:

(a) The limit of the denominator is zero.

(b) The limit of the numerator is zero.

SOLUTION Statement **(a)** is correct. The Quotient Law does not hold if the limit of the denominator is zero.

Exercises

In Exercises 1–24, evaluate the limit using the Basic Limit Laws and the limits $\lim_{x\to c} x^{p/q} = c^{p/q}$ *and* $\lim_{x\to c} k = k$.

1. $\lim_{x\to 9} x$

SOLUTION $\lim_{x\to 9} x = 9.$

2. $\lim_{x\to -3} 14$

SOLUTION $\lim_{x\to -3} 14 = 14.$

3. $\lim_{x\to \frac{1}{2}} x^4$

SOLUTION $\lim_{x\to \frac{1}{2}} x^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}.$

4. $\lim_{z\to 27} z^{2/3}$

SOLUTION $\lim_{z\to 27} z^{2/3} = 27^{2/3} = 9.$

5. $\lim_{t\to 2} t^{-1}$

SOLUTION $\lim_{t\to 2} t^{-1} = 2^{-1} = \frac{1}{2}.$

6. $\lim_{x\to 5} x^{-2}$

SOLUTION $\lim_{x\to 5} x^{-2} = 5^{-2} = \frac{1}{25}.$

7. $\lim_{x\to 0.2} (3x+4)$

SOLUTION Using the Sum Law and the Constant Multiple Law:

$$\begin{aligned}\lim_{x\to 0.2}(3x+4) &= \lim_{x\to 0.2} 3x + \lim_{x\to 0.2} 4\\ &= 3\lim_{x\to 0.2} x + \lim_{x\to 0.2} 4 = 3(0.2)+4 = 4.6.\end{aligned}$$

8. $\lim_{x\to \frac{1}{3}} (3x^3 + 2x^2)$

SOLUTION Using the Sum Law, the Constant Multiple Law and the Powers Law:

$$\begin{aligned}\lim_{x\to \frac{1}{3}}(3x^3+2x^2) &= \lim_{x\to \frac{1}{3}} 3x^3 + \lim_{x\to \frac{1}{3}} 2x^2\\ &= 3\lim_{x\to \frac{1}{3}} x^3 + 2\lim_{x\to \frac{1}{3}} x^2\\ &= 3\left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{3}\right)^2 = \frac{1}{3}.\end{aligned}$$

9. $\lim_{x\to -1} (3x^4 - 2x^3 + 4x)$

SOLUTION Using the Sum Law, the Constant Multiple Law and the Powers Law:

$$\begin{aligned}\lim_{x\to -1}(3x^4-2x^3+4x) &= \lim_{x\to -1} 3x^4 - \lim_{x\to -1} 2x^3 + \lim_{x\to -1} 4x\\ &= 3\lim_{x\to -1} x^4 - 2\lim_{x\to -1} x^3 + 4\lim_{x\to -1} x\\ &= 3(-1)^4 - 2(-1)^3 + 4(-1) = 3+2-4 = 1.\end{aligned}$$

10. $\lim_{x\to 8} (3x^{2/3} - 16x^{-1})$

SOLUTION Using the Sum Law, the Constant Multiple Law and the Powers Law:

$$\begin{aligned}\lim_{x\to 8}(3x^{2/3}-16x^{-1}) &= \lim_{x\to 8} 3x^{2/3} - \lim_{x\to 8} 16x^{-1}\\ &= 3\lim_{x\to 8} x^{2/3} - 16\lim_{x\to 8} x^{-1}\\ &= 3(8)^{2/3} - 16(8)^{-1} = 3(4) - 2 = 10.\end{aligned}$$

11. $\lim_{x\to 2} (x+1)(3x^2-9)$

SOLUTION Using the Product Law, the Sum Law and the Constant Multiple Law:

$$\begin{aligned}\lim_{x\to 2}(x+1)\left(3x^2-9\right) &= \left(\lim_{x\to 2} x + \lim_{x\to 2} 1\right)\left(\lim_{x\to 2} 3x^2 - \lim_{x\to 2} 9\right)\\ &= (2+1)\left(3\lim_{x\to 2} x^2 - 9\right)\\ &= 3(3(2)^2-9) = 9.\end{aligned}$$

12. $\lim_{x\to \frac{1}{2}} (4x+1)(6x-1)$

SOLUTION Using the Product Law, the Sum Law and the Constant Multiple Law:

$$\begin{aligned}\lim_{x\to 1/2}(4x+1)(6x-1) &= \left(\lim_{x\to 1/2}(4x+1)\right)\left(\lim_{x\to 1/2}(6x-1)\right)\\ &= \left(\lim_{x\to 1/2} 4x + \lim_{x\to 1/2} 1\right)\left(\lim_{x\to 1/2} 6x - \lim_{x\to 1/2} 1\right)\\ &= \left(4\lim_{x\to 1/2} x + \lim_{x\to 1/2} 1\right)\left(6\lim_{x\to 1/2} x - \lim_{x\to 1/2} 1\right)\\ &= \left(4\cdot\frac{1}{2}+1\right)\left(6\cdot\frac{1}{2}-1\right) = 3(2) = 6.\end{aligned}$$

13. $\displaystyle\lim_{t\to 4}\frac{3t-14}{t+1}$

SOLUTION Using the Quotient Law, the Sum Law and the Constant Multiple Law:

$$\lim_{t\to 4}\frac{3t-14}{t+1} = \frac{\lim_{t\to 4}(3t-14)}{\lim_{t\to 4}(t+1)} = \frac{3\lim_{t\to 4} t - \lim_{t\to 4} 14}{\lim_{t\to 4} t + \lim_{t\to 4} 1} = \frac{3\cdot 4 - 14}{4+1} = -\frac{2}{5}.$$

14. $\displaystyle\lim_{z\to 9}\frac{\sqrt{z}}{z-2}$

SOLUTION Using the Quotient Law, the Powers Law and the Sum Law:

$$\lim_{z\to 9}\frac{\sqrt{z}}{z-2} = \frac{\lim_{z\to 9}\sqrt{z}}{\lim_{z\to 9}(z-2)} = \frac{\lim_{z\to 9}\sqrt{z}}{\lim_{z\to 9} z - \lim_{z\to 9} 2} = \frac{3}{7}.$$

15. $\displaystyle\lim_{y\to\frac{1}{4}}(16y+1)(2y^{1/2}+1)$

SOLUTION Using the Product Law, the Sum Law, the Constant Multiple Law and the Powers Law:

$$\begin{aligned}\lim_{y\to\frac{1}{4}}(16y+1)(2y^{1/2}+1) &= \left(\lim_{y\to\frac{1}{4}}(16y+1)\right)\left(\lim_{y\to\frac{1}{4}}(2y^{1/2}+1)\right)\\ &= \left(16\lim_{y\to\frac{1}{4}} y + \lim_{y\to\frac{1}{4}} 1\right)\left(2\lim_{y\to\frac{1}{4}} y^{1/2} + \lim_{y\to\frac{1}{4}} 1\right)\\ &= \left(16\left(\frac{1}{4}\right)+1\right)\left(2\left(\frac{1}{2}\right)+1\right) = 10.\end{aligned}$$

16. $\displaystyle\lim_{x\to 2} x(x+1)(x+2)$

SOLUTION Using the Product Law and Sum Law:

$$\begin{aligned}\lim_{x\to 2} x(x+1)(x+2) &= \left(\lim_{x\to 2} x\right)\left(\lim_{x\to 2}(x+1)\right)\left(\lim_{x\to 2}(x+2)\right)\\ &= 2\left(\lim_{x\to 2} x + \lim_{x\to 2} 1\right)\left(\lim_{x\to 2} x + \lim_{x\to 2} 2\right)\\ &= 2(2+1)(2+2) = 24\end{aligned}$$

17. $\displaystyle\lim_{y\to 4}\frac{1}{\sqrt{6y+1}}$

SOLUTION Using the Quotient Law, the Powers Law, the Sum Law and the Constant Multiple Law:

$$\begin{aligned}\lim_{y\to 4}\frac{1}{\sqrt{6y+1}} &= \frac{1}{\lim_{y\to 4}\sqrt{6y+1}} = \frac{1}{\sqrt{6\lim_{y\to 4} y + 1}}\\ &= \frac{1}{\sqrt{6(4)+1}} = \frac{1}{5}.\end{aligned}$$

18. $\displaystyle\lim_{w\to 7}\frac{\sqrt{w+2}+1}{\sqrt{w-3}-1}$

SOLUTION Using the Quotient Law, the Sum Law and the Powers Law:

$$\lim_{w\to 7}\frac{\sqrt{w+2}+1}{\sqrt{w-3}-1} = \frac{\lim_{w\to 7}(\sqrt{w+2}+1)}{\lim_{w\to 7}(\sqrt{w-3}-1)} = \frac{\sqrt{\lim_{w\to 7}(w+2)}+1}{\sqrt{\lim_{w\to 7}(w-3)}-1} = \frac{\sqrt{9}+1}{\sqrt{4}-1} = 4.$$

19. $\lim_{x\to -1}\frac{x}{x^3+4x}$

SOLUTION Using the Quotient Law, the Sum Law, the Powers Law and the Constant Multiple Law:

$$\lim_{x\to -1}\frac{x}{x^3+4x} = \frac{\lim_{x\to -1} x}{\lim_{x\to -1} x^3 + 4\lim_{x\to -1} x} = \frac{-1}{(-1)^3+4(-1)} = \frac{1}{5}.$$

20. $\lim_{t\to -1}\frac{t^2+1}{(t^3+2)(t^4+1)}$

SOLUTION Using the Quotient Law, the Product Law, the Sum Law and the Powers Law:

$$\lim_{x\to -1}\frac{t^2+1}{(t^3+2)(t^4+1)} = \frac{\lim_{x\to -1} t^2 + \lim_{x\to -1} 1}{\left(\lim_{x\to -1} t^3 + \lim_{x\to -1} 2\right)\left(\lim_{x\to -1} t^4 + \lim_{x\to -1} 1\right)} = \frac{(-1)^2+1}{((-1)^3+2)((-1)^4+1)} = \frac{2}{(1)(2)} = 1.$$

21. $\lim_{t\to 25}\frac{3\sqrt{t}-\frac{1}{5}t}{(t-20)^2}$

SOLUTION Using the Quotient Law, the Sum Law, the Constant Multiple Law and the Powers Law:

$$\lim_{t\to 25}\frac{3\sqrt{t}-\frac{1}{5}t}{(t-20)^2} = \frac{3\sqrt{\lim_{t\to 25} t} - \frac{1}{5}\lim_{t\to 25} t}{\left(\lim_{t\to 25} t - 20\right)^2} = \frac{3(5)-\frac{1}{5}(25)}{5^2} = \frac{2}{5}.$$

22. $\lim_{y\to \frac{1}{3}}(18y^2-4)^4$

SOLUTION Using the Powers Law, the Sum Law and the Constant Multiple Law:

$$\lim_{y\to \frac{1}{3}}(18y^2-4)^4 = \left(18\lim_{y\to \frac{1}{3}} y^2 - 4\right)^4 = (2-4)^4 = 16.$$

23. $\lim_{t\to \frac{3}{2}}(4t^2+8t-5)^{3/2}$

SOLUTION Using the Powers Law, the Sum Law and the Constant Multiple Law:

$$\lim_{t\to \frac{3}{2}}(4t^2+8t-5)^{3/2} = \left(4\lim_{t\to \frac{3}{2}} t^2 + 8\lim_{t\to \frac{3}{2}} t - 5\right)^{3/2} = (9+12-5)^{3/2} = 64.$$

24. $\lim_{t\to 7}\frac{(t+2)^{1/2}}{(t+1)^{2/3}}$

SOLUTION Using the Quotient Law, the Powers Law and the Sum Law:

$$\lim_{t\to 7}\frac{(t+2)^{1/2}}{(t+1)^{2/3}} = \frac{\left(\lim_{t\to 7} t + 2\right)^{1/2}}{\left(\lim_{t\to 7} t + 1\right)^{2/3}} = \frac{9^{1/2}}{8^{2/3}} = \frac{3}{4}.$$

25. Use the Quotient Law to prove that if $\lim_{x\to c} f(x)$ exists and is nonzero, then

$$\lim_{x\to c} \frac{1}{f(x)} = \frac{1}{\lim_{x\to c} f(x)}$$

SOLUTION Since $\lim_{x\to c} f(x)$ is nonzero, we can apply the Quotient Law:

$$\lim_{x\to c} \left(\frac{1}{f(x)}\right) = \frac{\left(\lim_{x\to c} 1\right)}{\left(\lim_{x\to c} f(x)\right)} = \frac{1}{\lim_{x\to c} f(x)}.$$

26. Assuming that $\lim_{x\to 6} f(x) = 4$, compute:

(a) $\lim_{x\to 6} f(x)^2$ **(b)** $\lim_{x\to 6} \frac{1}{f(x)}$ **(c)** $\lim_{x\to 6} x\sqrt{f(x)}$

SOLUTION

(a) Using the Powers Law:

$$\lim_{x\to 6} f(x)^2 = \left(\lim_{x\to 6} f(x)\right)^2 = 4^2 = 16.$$

(b) Since $\lim_{x\to 6} f(x) \neq 0$, we may apply the Quotient Law:

$$\lim_{x\to 6} \frac{1}{f(x)} = \frac{1}{\lim_{x\to 6} f(x)} = \frac{1}{4}.$$

(c) Using the Product Law and Powers Law:

$$\lim_{x\to 6} x\sqrt{f(x)} = \left(\lim_{x\to 6} x\right)\left(\lim_{x\to 6} f(x)\right)^{1/2} = 6(4)^{1/2} = 12.$$

In Exercises 27–30, evaluate the limit assuming that $\lim_{x\to -4} f(x) = 3$ *and* $\lim_{x\to -4} g(x) = 1$.

27. $\lim_{x\to -4} f(x)g(x)$

SOLUTION $\lim_{x\to -4} f(x)g(x) = \lim_{x\to -4} f(x) \lim_{x\to -4} g(x) = 3\cdot 1 = 3.$

28. $\lim_{x\to -4} (2f(x) + 3g(x))$

SOLUTION

$$\begin{aligned}\lim_{x\to -4} (2f(x) + 3g(x)) &= 2\lim_{x\to -4} f(x) + 3\lim_{x\to -4} g(x)\\ &= 2\cdot 3 + 3\cdot 1 = 6 + 3 = 9.\end{aligned}$$

29. $\lim_{x\to -4} \frac{g(x)}{x^2}$

SOLUTION Since $\lim_{x\to -4} x^2 \neq 0$, we may apply the Quotient Law, then applying the Powers Law:

$$\lim_{x\to -4} \frac{g(x)}{x^2} = \frac{\lim_{x\to -4} g(x)}{\lim_{x\to -4} x^2} = \frac{1}{\left(\lim_{x\to -4} x\right)^2} = \frac{1}{16}.$$

30. $\lim_{x\to -4} \frac{f(x)+1}{3g(x)-9}$

SOLUTION

$$\lim_{x\to -4} \frac{f(x)+1}{3g(x)-9} = \frac{\lim_{x\to -4} f(x) + \lim_{x\to -4} 1}{3\lim_{x\to -4} g(x) - \lim_{x\to -4} 9} = \frac{3+1}{3\cdot 1 - 9} = \frac{4}{-6} = -\frac{2}{3}.$$

31. Can the Quotient Law be applied to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$? Explain.

SOLUTION The limit Quotient Law *cannot* be applied to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$ since $\lim_{x\to 0} x = 0$. This violates a condition of the Quotient Law. Accordingly, the rule *cannot* be employed.

32. Show that the Product Law cannot be used to evaluate the limit $\lim_{x\to \pi/2} \left(x - \frac{\pi}{2}\right) \tan x$.

SOLUTION The limit Product Law *cannot* be applied to evaluate $\lim_{x\to \pi/2} (x - \pi/2) \tan x$ since $\lim_{x\to \pi/2} \tan x$ does not exist (for example, as $x \to \pi/2-$, $\tan x \to \infty$). This violates a hypothesis of the Product Law. Accordingly, the rule *cannot* be employed.

33. Give an example where $\lim_{x\to 0} (f(x) + g(x))$ exists but neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists.

SOLUTION Let $f(x) = 1/x$ and $g(x) = -1/x$. Then $\lim_{x\to 0} (f(x) + g(x)) = \lim_{x\to 0} 0 = 0$ However, $\lim_{x\to 0} f(x) = \lim_{x\to 0} 1/x$ and $\lim_{x\to 0} g(x) = \lim_{x\to 0} -1/x$ do not exist.

Further Insights and Challenges

34. Show that if both $\lim_{x\to c} f(x)\,g(x)$ and $\lim_{x\to c} g(x)$ exist and $\lim_{x\to c} g(x) \neq 0$, then $\lim_{x\to c} f(x)$ exists. *Hint:* Write $f(x) = \frac{f(x)\,g(x)}{g(x)}$.

SOLUTION Given that $\lim_{x\to c} f(x)g(x) = L$ and $\lim_{x\to c} g(x) = M \neq 0$ both exist, observe that

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{f(x)g(x)}{g(x)} = \frac{\lim_{x\to c} f(x)g(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}$$

also exists.

35. Suppose that $\lim_{t\to 3} t g(t) = 12$. Show that $\lim_{t\to 3} g(t)$ exists and equals 4.

SOLUTION We are given that $\lim_{t\to 3} t g(t) = 12$. Since $\lim_{t\to 3} t = 3 \neq 0$, we may apply the Quotient Law:

$$\lim_{t\to 3} g(t) = \lim_{t\to 3} \frac{t g(t)}{t} = \frac{\lim_{t\to 3} t g(t)}{\lim_{t\to 3} t} = \frac{12}{3} = 4.$$

36. Prove that if $\lim_{t\to 3} \frac{h(t)}{t} = 5$, then $\lim_{t\to 3} h(t) = 15$.

SOLUTION Given that $\lim_{t\to 3} \frac{h(t)}{t} = 5$, observe that $\lim_{t\to 3} t = 3$. Now use the Product Law:

$$\lim_{t\to 3} h(t) = \lim_{t\to 3} t\,\frac{h(t)}{t} = \left(\lim_{t\to 3} t\right)\left(\lim_{t\to 3} \frac{h(t)}{t}\right) = 3 \cdot 5 = 15.$$

37. Assuming that $\lim_{x\to 0} \frac{f(x)}{x} = 1$, which of the following statements is necessarily true? Why?

(a) $f(0) = 0$

(b) $\lim_{x\to 0} f(x) = 0$

SOLUTION

(a) Given that $\lim_{x\to 0} \frac{f(x)}{x} = 1$, it is not necessarily true that $f(0) = 0$. A counterexample is provided by $f(x) = \begin{cases} x, & x \neq 0 \\ 5, & x = 0 \end{cases}$.

(b) Given that $\lim_{x\to 0} \frac{f(x)}{x} = 1$, it is necessarily true that $\lim_{x\to 0} f(x) = 0$. For note that $\lim_{x\to 0} x = 0$, whence

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\,\frac{f(x)}{x} = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} \frac{f(x)}{x}\right) = 0 \cdot 1 = 0.$$

38. Prove that if $\lim_{x\to c} f(x) = L \neq 0$ and $\lim_{x\to c} g(x) = 0$, then the limit $\lim_{x\to c} \frac{f(x)}{g(x)}$ does not exist.

SOLUTION Suppose that $\lim_{x\to c} \frac{f(x)}{g(x)}$ exists. Then

$$L = \lim_{x\to c} f(x) = \lim_{x\to c} g(x) \cdot \frac{f(x)}{g(x)} = \lim_{x\to c} g(x) \cdot \lim_{x\to c} \frac{f(x)}{g(x)} = 0 \cdot \lim_{x\to c} \frac{f(x)}{g(x)} = 0.$$

But, we were given that $L \neq 0$, so we have arrived at a contradiction. Thus, $\lim_{x\to c} \frac{f(x)}{g(x)}$ does not exist.

39. Suppose that $\lim_{h\to 0} g(h) = L$.

(a) Explain why $\lim_{h\to 0} g(ah) = L$ for any constant $a \neq 0$.

(b) If we assume instead that $\lim_{h\to 1} g(h) = L$, is it still necessarily true that $\lim_{h\to 1} g(ah) = L$?

(c) Illustrate (a) and (b) with the function $f(x) = x^2$.

SOLUTION

(a) As $h \to 0$, $ah \to 0$ as well; hence, if we make the change of variable $w = ah$, then

$$\lim_{h\to 0} g(ah) = \lim_{w\to 0} g(w) = L.$$

(b) No. As $h \to 1$, $ah \to a$, so we should not expect $\lim_{h\to 1} g(ah) = \lim_{h\to 1} g(h)$.

(c) Let $g(x) = x^2$. Then

$$\lim_{h\to 0} g(h) = 0 \quad \text{and} \quad \lim_{h\to 0} g(ah) = \lim_{h\to 0} (ah)^2 = 0.$$

On the other hand,

$$\lim_{h\to 1} g(h) = 1 \quad \text{while} \quad \lim_{h\to 1} g(ah) = \lim_{h\to 1} (ah)^2 = a^2,$$

which is equal to the previous limit if and only if $a = \pm 1$.

40. Assume that $L(a) = \lim_{x\to 0} \dfrac{a^x - 1}{x}$ exists for all $a > 0$. Assume also that $\lim_{x\to 0} a^x = 1$.

(a) Prove that $L(ab) = L(a) + L(b)$ for $a, b > 0$. *Hint:* $(ab)^x - 1 = a^x(b^x - 1) + (a^x - 1)$. This shows that $L(a)$ "behaves" like a logarithm. We will see that $L(a) = \ln a$ in Section 3.10.

(b) Verify numerically that $L(12) = L(3) + L(4)$.

SOLUTION

(a) Let $a, b > 0$. Then

$$\begin{aligned} L(ab) = \lim_{x\to 0} \frac{(ab)^x - 1}{x} &= \lim_{x\to 0} \frac{a^x(b^x - 1) + (a^x - 1)}{x} \\ &= \lim_{x\to 0} a^x \cdot \lim_{x\to 0} \frac{b^x - 1}{x} + \lim_{x\to 0} \frac{a^x - 1}{x} \\ &= 1 \cdot L(b) + L(a) = L(a) + L(b). \end{aligned}$$

(b) From the table below, we estimate that, to three decimal places, $L(3) = 1.099$, $L(4) = 1.386$ and $L(12) = 2.485$. Thus,

$$L(12) = 2.485 = 1.099 + 1.386 = L(3) + L(4).$$

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$(3^x - 1)/x$	1.092600	1.098009	1.098552	1.098673	1.099216	1.104669
$(4^x - 1)/x$	1.376730	1.385334	1.386198	1.386390	1.387256	1.395948
$(12^x - 1)/x$	2.454287	2.481822	2.484600	2.485215	2.488000	2.516038

2.4 Limits and Continuity

Preliminary Questions

1. Which property of $f(x) = x^3$ allows us to conclude that $\lim_{x\to 2} x^3 = 8$?

SOLUTION We can conclude that $\lim_{x\to 2} x^3 = 8$ because the function x^3 is continuous at $x = 2$.

2. What can be said about $f(3)$ if f is continuous and $\lim_{x\to 3} f(x) = \frac{1}{2}$?

SOLUTION If f is continuous and $\lim_{x\to 3} f(x) = \frac{1}{2}$, then $f(3) = \frac{1}{2}$.

3. Suppose that $f(x) < 0$ if x is positive and $f(x) > 1$ if x is negative. Can f be continuous at $x = 0$?

SOLUTION Since $f(x) < 0$ when x is positive and $f(x) > 1$ when x is negative, it follows that

$$\lim_{x\to 0+} f(x) \le 0 \quad \text{and} \quad \lim_{x\to 0-} f(x) \ge 1.$$

Thus, $\lim_{x\to 0} f(x)$ does not exist, so f cannot be continuous at $x = 0$.

4. Is it possible to determine $f(7)$ if $f(x) = 3$ for all $x < 7$ and f is right-continuous at $x = 7$? What if f is left-continuous?

SOLUTION No. To determine $f(7)$, we need to combine either knowledge of the values of $f(x)$ for $x < 7$ with *left*-continuity or knowledge of the values of $f(x)$ for $x > 7$ with right-continuity.

5. Are the following true or false? If false, state a correct version.

(a) $f(x)$ is continuous at $x = a$ if the left- and right-hand limits of $f(x)$ as $x \to a$ exist and are equal.

(b) $f(x)$ is continuous at $x = a$ if the left- and right-hand limits of $f(x)$ as $x \to a$ exist and equal $f(a)$.

(c) If the left- and right-hand limits of $f(x)$ as $x \to a$ exist, then f has a removable discontinuity at $x = a$.

(d) If $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x) + g(x)$ is continuous at $x = a$.

(e) If $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x)/g(x)$ is continuous at $x = a$.

SOLUTION

(a) False. The correct statement is "$f(x)$ is continuous at $x = a$ if the left- and right-hand limits of $f(x)$ as $x \to a$ exist and equal $f(a)$."

(b) True.

(c) False. The correct statement is "If the left- and right-hand limits of $f(x)$ as $x \to a$ are equal but not equal to $f(a)$, then f has a removable discontinuity at $x = a$."

(d) True.

(e) False. The correct statement is "If $f(x)$ and $g(x)$ are continuous at $x = a$ and $g(a) \neq 0$, then $f(x)/g(x)$ is continuous at $x = a$."

Exercises

1. Referring to Figure 1, state whether $f(x)$ is left- or right-continuous (or neither) at each point of discontinuity. Does $f(x)$ have any removable discontinuities?

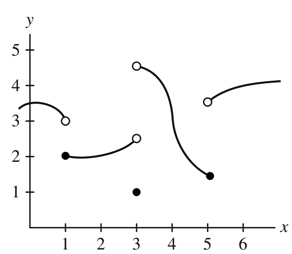


FIGURE 1 Graph of $y = f(x)$

SOLUTION

- The function f is discontinuous at $x = 1$; it is right-continuous there.
- The function f is discontinuous at $x = 3$; it is neither left-continuous nor right-continuous there.
- The function f is discontinuous at $x = 5$; it is left-continuous there.

However, these discontinuities are not removable.

Exercises 2–4 refer to the function $g(x)$ in Figure 2.

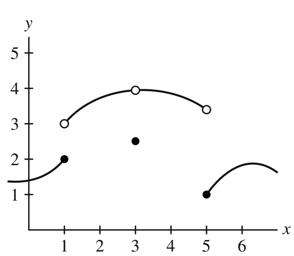


FIGURE 2 Graph of $y = g(x)$

2. State whether $g(x)$ is left- or right-continuous (or neither) at each of its points of discontinuity.

SOLUTION

- The function g is discontinuous at $x = 1$; it is left-continuous there.
- The function g is discontinuous at $x = 3$; it is neither left-continuous nor right-continuous there.
- The function g is discontinuous at $x = 5$; it is right-continuous there.

3. At which point c does $g(x)$ have a removable discontinuity? How should $g(c)$ be redefined to make g continuous at $x = c$?

SOLUTION Because $\lim_{x\to 3} g(x)$ exists, the function g has a removable discontinuity at $x = 3$. Assigning $g(3) = 4$ makes g continuous at $x = 3$.

4. Find the point c_1 at which $g(x)$ has a jump discontinuity but is left-continuous. How should $g(c_1)$ be redefined to make g right-continuous at $x = c_1$?

SOLUTION The function g has a jump discontinuity at $x = 1$, but is left-continuous there. Assigning $g(1) = 3$ makes g right-continuous at $x = 1$ (but no longer left-continuous).

5. In Figure 3, determine the one-sided limits at the points of discontinuity. Which discontinuity is removable and how should f be redefined to make it continuous at this point?

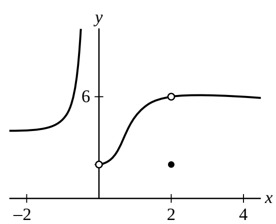


FIGURE 3

SOLUTION The function f is discontinuous at $x = 0$, at which $\lim_{x\to 0-} f(x) = \infty$ and $\lim_{x\to 0+} f(x) = 2$. The function f is also discontinuous at $x = 2$, at which $\lim_{x\to 2-} f(x) = 6$ and $\lim_{x\to 2+} f(x) = 6$. Because the two one-sided limits exist and are equal at $x = 2$, the discontinuity at $x = 2$ is removable. Assigning $f(2) = 6$ makes f continuous at $x = 2$.

6. Suppose that $f(x) = 2$ for $x < 3$ and $f(x) = -4$ for $x > 3$.

(a) What is $f(3)$ if f is left-continuous at $x = 3$?

(b) What is $f(3)$ if f is right-continuous at $x = 3$?

SOLUTION $f(x) = 2$ for $x < 3$ and $f(x) = -4$ for $x > 3$.

- If f is left-continuous at $x = 3$, then $f(3) = \lim_{x\to 3-} f(x) = 2$.
- If f is right-continuous at $x = 3$, then $f(3) = \lim_{x\to 0+} f(x) = -4$.

In Exercises 7–16, use the Laws of Continuity and Theorems 2 and 3 to show that the function is continuous.

7. $f(x) = x + \sin x$

SOLUTION Since x and $\sin x$ are continuous, so is $x + \sin x$ by Continuity Law (i).

8. $f(x) = x \sin x$

SOLUTION Since x and $\sin x$ are continuous, so is $x \sin x$ by Continuity Law (iii).

9. $f(x) = 3x + 4 \sin x$

SOLUTION Since x and $\sin x$ are continuous, so are $3x$ and $4 \sin x$ by Continuity Law (ii). Thus $3x + 4 \sin x$ is continuous by Continuity Law (i).

10. $f(x) = 3x^3 + 8x^2 - 20x$

SOLUTION

- Since x is continuous, so are x^3 and x^2 by repeated applications of Continuity Law (iii).
- Hence $3x^3$, $8x^2$, and $-20x$ are continuous by Continuity Law (ii).
- Finally, $3x^3 + 8x^2 - 20x$ is continuous by Continuity Law (i).

11. $f(x) = \dfrac{1}{x^2 + 1}$

SOLUTION

- Since x is continuous, so is x^2 by Continuity Law (iii).
- Recall that constant functions, such as 1, are continuous. Thus $x^2 + 1$ is continuous.
- Finally, $\dfrac{1}{x^2+1}$ is continuous by Continuity Law (iv) because $x^2 + 1$ is never 0.

12. $f(x) = \dfrac{x^2 - \cos x}{3 + \cos x}$

SOLUTION

- Since x is continuous, so is x^2 by Continuity Law (iii).
- Since $\cos x$ is continuous, so is $-\cos x$ by Continuity Law (ii).
- Accordingly, $x^2 - \cos x$ is continuous by Continuity Law (i).
- Since 3 (a constant function) and $\cos x$ are continuous, so is $3 + \cos x$ by Continuity Law (i).
- Finally, $\dfrac{x^2 - \cos x}{3 + \cos x}$ is continuous by Continuity Law (iv) because $3 + \cos x$ is never 0.

13. $f(x) = \cos(x^2)$

SOLUTION The function $f(x)$ is a composite of two continuous functions: $\cos x$ and x^2, so $f(x)$ is continuous by Theorem 5, which states that a composite of continuous functions is continuous.

14. $f(x) = \tan^{-1}(4^x)$

SOLUTION The function $f(x)$ is a composite of two continuous functions: $\tan^{-1} x$ and 4^x, so $f(x)$ is continuous by Theorem 5, which states that a composite of continuous functions is continuous.

15. $f(x) = e^x \cos 3x$

SOLUTION e^x and $\cos 3x$ are continuous, so $e^x \cos 3x$ is continuous by Continuity Law (iii).

16. $f(x) = \ln(x^4 + 1)$

SOLUTION

- Since x is continuous, so is x^4 by repeated application of Continuity Law (iii).
- Since 1 (a constant function) and x^4 are continuous, so is $x^4 + 1$ by Continuity Law (i).
- Finally, because $x^4 + 1 > 0$ for all x and $\ln x$ is continuous for $x > 0$, the composite function $\ln(x^4 + 1)$ is continuous.

In Exercises 17–34, determine the points of discontinuity. State the type of discontinuity (removable, jump, infinite, or none of these) and whether the function is left- or right-continuous.

17. $f(x) = \dfrac{1}{x}$

SOLUTION The function $1/x$ is discontinuous at $x = 0$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $x = 0$.

18. $f(x) = |x|$

SOLUTION The function $f(x) = |x|$ is continuous everywhere.

19. $f(x) = \dfrac{x-2}{|x-1|}$

SOLUTION The function $\dfrac{x-2}{|x-1|}$ is discontinuous at $x = 1$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $x = 1$.

20. $f(x) = [x]$

SOLUTION This function has a jump discontinuity at $x = n$ for every integer n. It is continuous at all other values of x. For every integer n,

$$\lim_{x \to n+} [x] = n$$

since $[x] = n$ for all x between n and $n + 1$. This shows that $[x]$ is *right-continuous* at $x = n$. On the other hand,

$$\lim_{x \to n-} [x] = n - 1$$

since $[x] = n - 1$ for all x between $n - 1$ and n. Thus $[x]$ is not left-continuous.

21. $f(x) = \left[\frac{1}{2}x\right]$

SOLUTION The function $\left[\frac{1}{2}x\right]$ is discontinuous at even integers, at which there are jump discontinuities. Because

$$\lim_{x\to 2n+} \left[\frac{1}{2}x\right] = n$$

but

$$\lim_{x\to 2n-} \left[\frac{1}{2}x\right] = n - 1,$$

it follows that this function is right-continuous at the even integers but not left-continuous.

22. $g(t) = \dfrac{1}{t^2 - 1}$

SOLUTION The function $f(t) = \dfrac{1}{t^2-1} = \dfrac{1}{(t-1)(t+1)}$ is discontinuous at $t = -1$ and $t = 1$, at which there are infinite discontinuities. The function is neither left- nor right- continuous at either point of discontinuity.

23. $f(x) = \dfrac{x+1}{4x-2}$

SOLUTION The function $f(x) = \dfrac{x+1}{4x-2}$ is discontinuous at $x = \frac{1}{2}$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $x = \frac{1}{2}$.

24. $h(z) = \dfrac{1-2z}{z^2 - z - 6}$

SOLUTION The function $f(z) = \dfrac{1-2z}{z^2-z-6} = \dfrac{1-2z}{(z+2)(z-3)}$ is discontinuous at $z = -2$ and $z = 3$, at which there are infinite discontinuities. The function is neither left- nor right- continuous at either point of discontinuity.

25. $f(x) = 3x^{2/3} - 9x^3$

SOLUTION The function $f(x) = 3x^{2/3} - 9x^3$ is defined and continuous for all x.

26. $g(t) = 3t^{-2/3} - 9t^3$

SOLUTION The function $g(t) = 3t^{-2/3} - 9t^3$ is discontinuous at $t = 0$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $t = 0$.

27. $f(x) = \begin{cases} \dfrac{x-2}{|x-2|} & x \neq 2 \\ -1 & x = 2 \end{cases}$

SOLUTION For $x > 2$, $f(x) = \dfrac{x-2}{(x-2)} = 1$. For $x < 2$, $f(x) = \dfrac{(x-2)}{(2-x)} = -1$. The function has a jump discontinuity at $x = 2$. Because

$$\lim_{x\to 2-} f(x) = -1 = f(2)$$

but

$$\lim_{x\to 2+} f(x) = 1 \neq f(2),$$

it follows that this function is left-continuous at $x = 2$ but not right-continuous.

28. $f(x) = \begin{cases} \cos\dfrac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

SOLUTION The function $\cos\left(\frac{1}{x}\right)$ is discontinuous at $x = 0$, at which there is an oscillatory discontinuity. Because neither

$$\lim_{x\to 0-} f(x) \quad \text{nor} \quad \lim_{x\to 0+} f(x)$$

exist, the function is neither left- nor right-continuous at $x = 0$.

29. $g(t) = \tan 2t$

SOLUTION The function $g(t) = \tan 2t = \dfrac{\sin 2t}{\cos 2t}$ is discontinuous whenever $\cos 2t = 0$; i.e., whenever

$$2t = \frac{(2n+1)\pi}{2} \qquad \text{or} \qquad t = \frac{(2n+1)\pi}{4},$$

where n is an integer. At every such value of t there is an infinite discontinuity. The function is neither left- nor right-continuous at any of these points of discontinuity.

30. $f(x) = \csc(x^2)$

SOLUTION The function $f(x) = \csc(x^2) = \dfrac{1}{\sin(x^2)}$ is discontinuous whenever $\sin(x^2) = 0$; i.e., whenever $x^2 = n\pi$ or $x = \pm\sqrt{n\pi}$, where n is a positive integer. At every such value of x there is an infinite discontinuity. The function is neither left- nor right-continuous at any of these points of discontinuity.

31. $f(x) = \tan(\sin x)$

SOLUTION The function $f(x) = \tan(\sin x)$ is continuous everywhere. Reason: $\sin x$ is continuous everywhere and $\tan u$ is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$—and in particular on $-1 \le u = \sin x \le 1$. Continuity of $\tan(\sin x)$ follows by the continuity of composite functions.

32. $f(x) = \cos(\pi[x])$

SOLUTION The function $f(x) = \cos(\pi[x])$ has a jump discontinuity at $x = n$ for every integer n. The function is right-continuous but not left-continuous at each of these points of discontinuity.

33. $f(x) = \dfrac{1}{e^x - e^{-x}}$

SOLUTION The function $f(x) = \dfrac{1}{e^x - e^{-x}}$ is discontinuous at $x = 0$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $x = 0$.

34. $f(x) = \ln|x - 4|$

SOLUTION The function $f(x) = \ln|x - 4|$ is discontinuous at $x = 4$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $x = 4$.

In Exercises 35–48, determine the domain of the function and prove that it is continuous on its domain using the Laws of Continuity and the facts quoted in this section.

35. $f(x) = 2\sin x + 3\cos x$

SOLUTION The domain of $2\sin x + 3\cos x$ is all real numbers. Both $\sin x$ and $\cos x$ are continuous on this domain, so $2\sin x + 3\cos x$ is continuous by Continuity Laws (i) and (ii).

36. $f(x) = \sqrt{x^2 + 9}$

SOLUTION The domain of $\sqrt{x^2 + 9}$ is all real numbers, as $x^2 + 9 > 0$ for all x. Since $\sqrt{x}$ and the polynomial $x^2 + 9$ are both continuous, so is the composite function $\sqrt{x^2 + 9}$.

37. $f(x) = \sqrt{x}\sin x$

SOLUTION This function is defined as long as $x \ge 0$. Since $\sqrt{x}$ and $\sin x$ are continuous, so is $\sqrt{x}\sin x$ by Continuity Law (iii).

38. $f(x) = \dfrac{x^2}{x + x^{1/4}}$

SOLUTION This function is defined as long as $x \ge 0$ and $x + x^{1/4} \ne 0$, and so the domain is all $x > 0$. Since x is continuous, so are x^2 and $x + x^{1/4}$ by Continuity Laws (iii) and (i); hence, by Continuity Law (iv), so is $\dfrac{x^2}{x + x^{1/4}}$.

39. $f(x) = x^{2/3}2^x$

SOLUTION The domain of $x^{2/3}2^x$ is all real numbers as the denominator of the rational exponent is odd. Both $x^{2/3}$ and 2^x are continuous on this domain, so $x^{2/3}2^x$ is continuous by Continuity Law (iii).

40. $f(x) = x^{1/3} + x^{3/4}$

SOLUTION The domain of $x^{1/3} + x^{3/4}$ is $x \ge 0$. On this domain, both $x^{1/3}$ and $x^{3/4}$ are continuous, so $x^{1/3} + x^{3/4}$ is continuous by Continuity Law (i).

41. $f(x) = x^{-4/3}$

SOLUTION This function is defined for all $x \ne 0$. Because the function $x^{4/3}$ is continuous and not equal to zero for $x \ne 0$, it follows that

$$x^{-4/3} = \frac{1}{x^{4/3}}$$

is continuous for $x \ne 0$ by Continuity Law (iv).

42. $f(x) = \ln(9 - x^2)$

SOLUTION The domain of $\ln(9 - x^2)$ is all x such that $9 - x^2 > 0$, or $|x| < 3$. The polynomial $9 - x^2$ is continuous for all real numbers and $\ln x$ is continuous for $x > 0$; therefore, the composite function $\ln(9 - x^2)$ is continuous for $|x| < 3$.

43. $f(x) = \tan^2 x$

SOLUTION The domain of $\tan^2 x$ is all $x \neq \pm(2n-1)\pi/2$ where n is a positive integer. Because $\tan x$ is continuous on this domain, it follows from Continuity Law (iii) that $\tan^2 x$ is also continuous on this domain.

44. $f(x) = \cos(2^x)$

SOLUTION The domain of $\cos(2^x)$ is all real numbers. Because the functions $\cos x$ and 2^x are continuous on this domain, so is the composite function $\cos(2^x)$.

45. $f(x) = (x^4+1)^{3/2}$

SOLUTION The domain of $(x^4+1)^{3/2}$ is all real numbers as $x^4+1>0$ for all x. Because $x^{3/2}$ and the polynomial x^4+1 are both continuous, so is the composite function $(x^4+1)^{3/2}$.

46. $f(x) = e^{-x^2}$

SOLUTION The domain of e^{-x^2} is all real numbers. Because e^x and the polynomial $-x^2$ are both continuous for all real numbers, so is the composite function e^{-x^2}.

47. $f(x) = \dfrac{\cos(x^2)}{x^2-1}$

SOLUTION The domain for this function is all $x \neq \pm 1$. Because the functions $\cos x$ and x^2 are continuous on this domain, so is the composite function $\cos(x^2)$. Finally, because the polynomial x^2-1 is continuous and not equal to zero for $x \neq \pm 1$, the function $\dfrac{\cos(x^2)}{x^2-1}$ is continuous by Continuity Law (iv).

48. $f(x) = 9^{\tan x}$

SOLUTION The domain of $9^{\tan x}$ is all $x \neq \pm(2n-1)\pi/2$ where n is a positive integer. Because $\tan x$ and 9^x are continuous on this domain, it follows that the composite function $9^{\tan x}$ is also continuous on this domain.

49. Show that the function

$$f(x) = \begin{cases} x^2+3 & \text{for } x < 1 \\ 10-x & \text{for } 1 \le x \le 2 \\ 6x - x^2 & \text{for } x > 2 \end{cases}$$

is continuous for $x \neq 1, 2$. Then compute the right- and left-hand limits at $x = 1, 2$, and determine whether $f(x)$ is left-continuous, right-continuous, or continuous at these points (Figure 4).

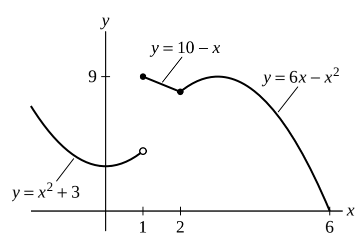


FIGURE 4

SOLUTION Let's start with $x \neq 1, 2$.

- Because x is continuous, so is x^2 by Continuity Law (iii). The constant function 3 is also continuous, so x^2+3 is continuous by Continuity Law (i). Therefore, $f(x)$ is continuous for $x < 1$.
- Because x and the constant function 10 are continuous, the function $10-x$ is continuous by Continuity Law (i). Therefore, $f(x)$ is continuous for $1 < x < 2$.
- Because x is continuous, x^2 is continuous by Continuity Law (iii) and $6x$ is continuous by Continuity Law (ii). Therefore, $6x-x^2$ is continuous by Continuity Law (i), so $f(x)$ is continuous for $x > 2$.

At $x = 1$, $f(x)$ has a jump discontinuity because the one-sided limits exist but are not equal:

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} (x^2+3) = 4, \qquad \lim_{x\to 1+} f(x) = \lim_{x\to 1+} (10-x) = 9.$$

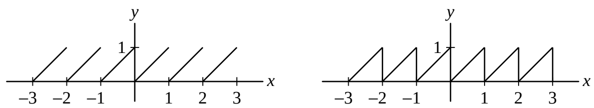
Furthermore, the right-hand limit equals the function value $f(1) = 9$, so $f(x)$ is right-continuous at $x = 1$. At $x = 2$,

$$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} (10-x) = 8, \qquad \lim_{x\to 2+} f(x) = \lim_{x\to 2+} (6x-x^2) = 8.$$

The left- and right-hand limits exist and are equal to $f(2)$, so $f(x)$ is continuous at $x = 2$.

50. Sawtooth Function Draw the graph of $f(x) = x - [x]$. At which points is f discontinuous? Is it left- or right-continuous at those points?

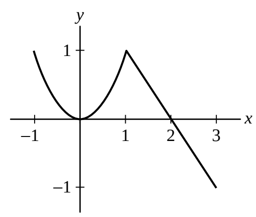
SOLUTION Two views of the sawtooth function $f(x) = x - [x]$ appear below. The first is the actual graph. In the second, the jumps are "connected" so as to better illustrate its "sawtooth" nature. The function is right-continuous at integer values of x.



In Exercises 51–54, sketch the graph of $f(x)$. At each point of discontinuity, state whether f is left- or right-continuous.

51. $f(x) = \begin{cases} x^2 & \text{for } x \le 1 \\ 2 - x & \text{for } x > 1 \end{cases}$

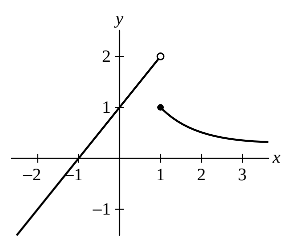
SOLUTION



The function f is continuous everywhere.

52. $f(x) = \begin{cases} x + 1 & \text{for } x < 1 \\ \dfrac{1}{x} & \text{for } x \ge 1 \end{cases}$

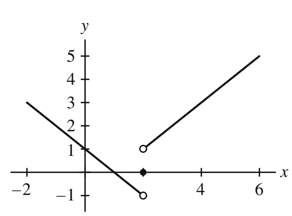
SOLUTION



The function f is right-continuous at $x = 1$.

53. $f(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{|x - 2|} & x \ne 2 \\ 0 & x = 2 \end{cases}$

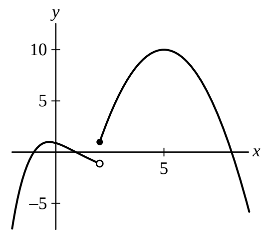
SOLUTION



The function f is neither left- nor right-continuous at $x = 2$.

54. $f(x) = \begin{cases} x^3 + 1 & \text{for } -\infty < x \le 0 \\ -x + 1 & \text{for } 0 < x < 2 \\ -x^2 + 10x - 15 & \text{for } x \ge 2 \end{cases}$

SOLUTION



The function f is right-continuous at $x = 2$.

55. Show that the function

$$f(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4} & x \neq 4 \\ 10 & x = 4 \end{cases}$$

has a removable discontinuity at $x = 4$.

SOLUTION To show that $f(x)$ has a removable discontinuity at $x = 4$, we must establish that

$$\lim_{x \to 4} f(x)$$

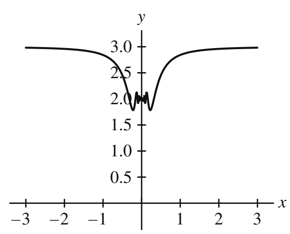
exists but does not equal $f(4)$. Now,

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} (x + 4) = 8 \neq 10 = f(4);$$

thus, $f(x)$ has a removable discontinuity at $x = 4$. To remove the discontinuity, we must redefine $f(4) = 8$.

56. GU Define $f(x) = x \sin \frac{1}{x} + 2$ for $x \neq 0$. Plot $f(x)$. How should $f(0)$ be defined so that f is continuous at $x = 0$?

SOLUTION



From the graph, it appears that $f(0)$ should be defined equal to 2 to make f continuous at $x = 0$.

In Exercises 57–59, find the value of the constant (a, b, or c) that makes the function continuous.

57. $f(x) = \begin{cases} x^2 - c & \text{for } x < 5 \\ 4x + 2c & \text{for } x \geq 5 \end{cases}$

SOLUTION As $x \to 5-$, we have $x^2 - c \to 25 - c = L$. As $x \to 5+$, we have $4x + 2c \to 20 + 2c = R$. Match the limits: $L = R$ or $25 - c = 20 + 2c$ implies $c = \frac{5}{3}$.

58. $f(x) = \begin{cases} 2x + 9x^{-1} & \text{for } x \leq 3 \\ -4x + c & \text{for } x > 3 \end{cases}$

SOLUTION As $x \to 3-$, we have $2x + 9x^{-1} \to 9 = L$. As $x \to 3+$, we have $-4x + c \to c - 12 = R$. Match the limits: $L = R$ or $9 = c - 12$ implies $c = 21$.

59. $f(x) = \begin{cases} x^{-1} & \text{for } x < -1 \\ ax + b & \text{for } -1 \leq x \leq \frac{1}{2} \\ x^{-1} & \text{for } x > \frac{1}{2} \end{cases}$

SOLUTION As $x \to -1-$, $x^{-1} \to -1$ while as $x \to -1+$, $ax + b \to b - a$. For f to be continuous at $x = -1$, we must therefore have $b - a = -1$. Now, as $x \to \frac{1}{2}-$, $ax + b \to \frac{1}{2}a + b$ while as $x \to \frac{1}{2}+$, $x^{-1} \to 2$. For f to be continuous at $x = \frac{1}{2}$, we must therefore have $\frac{1}{2}a + b = 2$. Solving these two equations for a and b yields $a = 2$ and $b = 1$.

60. Define

$$g(x) = \begin{cases} x + 3 & \text{for } x < -1 \\ cx & \text{for } -1 \le x \le 2 \\ x + 2 & \text{for } x > 2 \end{cases}$$

Find a value of c such that $g(x)$ is

(a) left-continuous

(b) right-continuous

In each case, sketch the graph of $g(x)$.

SOLUTION

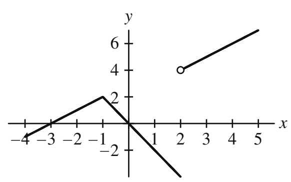
(a) In order for $g(x)$ to be left-continuous, we need

$$\lim_{x\to -1-} g(x) = \lim_{x\to -1-} (x + 3) = 2$$

to be equal to

$$\lim_{x\to -1+} g(x) = \lim_{x\to -1+} cx = -c.$$

Therefore, we must have $c = -2$. The graph of $g(x)$ with $c = -2$ is shown below.



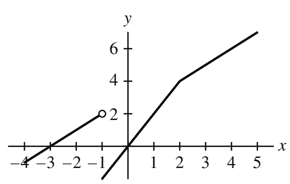
(b) In order for $g(x)$ to be right-continuous, we need

$$\lim_{x\to 2-} g(x) = \lim_{x\to 2-} cx = 2c$$

to be equal to

$$\lim_{x\to 2+} g(x) = \lim_{x\to 2+} (x + 2) = 4.$$

Therefore, we must have $c = 2$. The graph of $g(x)$ with $c = 2$ is shown below.



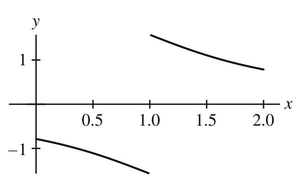
61. Define $g(t) = \tan^{-1}\left(\dfrac{1}{t-1}\right)$ for $t \neq 1$. Answer the following questions, using a plot if necessary.

(a) Can $g(1)$ be defined so that $g(t)$ is continuous at $t = 1$?

(b) How should $g(1)$ be defined so that $g(t)$ is left-continuous at $t = 1$?

SOLUTION

(a) From the graph of $g(t)$ shown below, we see that g has a jump discontinuity at $t = 1$; therefore, $g(a)$ cannot be defined so that g is continuous at $t = 1$.



(b) To make g left-continuous at $t = 1$, we should define

$$g(1) = \lim_{t\to 1-} \tan^{-1}\left(\tfrac{1}{t-1}\right) = -\frac{\pi}{2}.$$

62. Each of the following statements is *false*. For each statement, sketch the graph of a function that provides a counterexample.

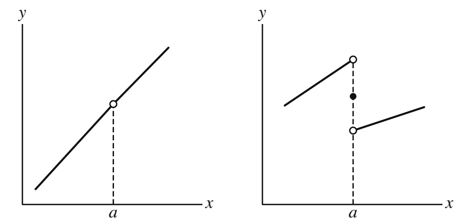
(a) If $\lim_{x\to a} f(x)$ exists, then $f(x)$ is continuous at $x = a$.

(b) If $f(x)$ has a jump discontinuity at $x = a$, then $f(a)$ is equal to either $\lim_{x\to a-} f(x)$ or $\lim_{x\to a+} f(x)$.

SOLUTION Refer to the two figures shown below.

(a) The figure at the left shows a function for which $\lim_{x\to a} f(x)$ exists, but the function is not continuous at $x = a$ because the function is not defined at $x = a$.

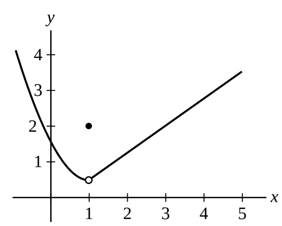
(b) The figure at the right shows a function that has a jump discontinuity at $x = a$ but $f(a)$ is not equal to either $\lim_{x\to a-} f(x)$ or $\lim_{x\to a+} f(x)$.



In Exercises 63–66, draw the graph of a function on $[0, 5]$ *with the given properties.*

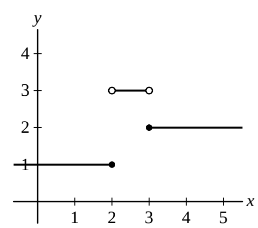
63. $f(x)$ is not continuous at $x = 1$, but $\lim_{x\to 1+} f(x)$ and $\lim_{x\to 1-} f(x)$ exist and are equal.

SOLUTION



64. $f(x)$ is left-continuous but not continuous at $x = 2$ and right-continuous but not continuous at $x = 3$.

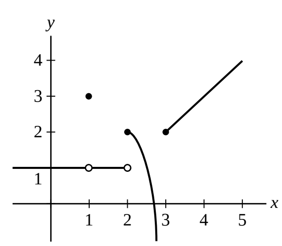
SOLUTION



65. $f(x)$ has a removable discontinuity at $x = 1$, a jump discontinuity at $x = 2$, and

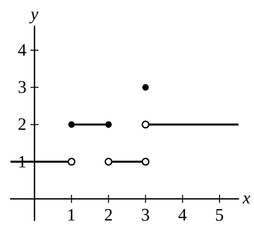
$$\lim_{x\to 3-} f(x) = -\infty, \qquad \lim_{x\to 3+} f(x) = 2$$

SOLUTION



66. $f(x)$ is right- but not left-continuous at $x = 1$, left- but not right-continuous at $x = 2$, and neither left- nor right-continuous at $x = 3$.

SOLUTION



In Exercises 67–80, evaluate using substitution.

67. $\lim_{x\to -1} (2x^3 - 4)$

SOLUTION $\lim_{x\to -1} (2x^3 - 4) = 2(-1)^3 - 4 = -6.$

68. $\lim_{x\to 2} (5x - 12x^{-2})$

SOLUTION $\lim_{x\to 2} (5x - 12x^{-2}) = 5(2) - 12(2^{-2}) = 10 - 12(\frac{1}{4}) = 7.$

69. $\lim_{x\to 3} \frac{x+2}{x^2+2x}$

SOLUTION $\lim_{x\to 3} \frac{x+2}{x^2+2x} = \frac{3+2}{3^2+2\cdot 3} = \frac{5}{15} = \frac{1}{3}$

70. $\lim_{x\to \pi} \sin\left(\frac{x}{2} - \pi\right)$

SOLUTION $\lim_{x\to \pi} \sin(\frac{x}{2} - \pi) = \sin(-\frac{\pi}{2}) = -1.$

71. $\lim_{x\to \frac{\pi}{4}} \tan(3x)$

SOLUTION $\lim_{x\to \frac{\pi}{4}} \tan(3x) = \tan(3\cdot\frac{\pi}{4}) = \tan(\frac{3\pi}{4}) = -1$

72. $\lim_{x\to \pi} \frac{1}{\cos x}$

SOLUTION $\lim_{x\to \pi} \frac{1}{\cos x} = \frac{1}{\cos \pi} = \frac{1}{-1} = -1.$

73. $\lim_{x\to 4} x^{-5/2}$

SOLUTION $\lim_{x\to 4} x^{-5/2} = 4^{-5/2} = \frac{1}{32}.$

74. $\lim_{x\to 2} \sqrt{x^3 + 4x}$

SOLUTION $\lim_{x\to 2} \sqrt{x^3 + 4x} = \sqrt{2^3 + 4(2)} = 4.$

75. $\lim_{x\to -1} (1 - 8x^3)^{3/2}$

SOLUTION $\lim_{x\to -1} (1 - 8x^3)^{3/2} = (1 - 8(-1)^3)^{3/2} = 27.$

76. $\lim_{x\to 2} \left(\frac{7x+2}{4-x}\right)^{2/3}$

SOLUTION $\lim_{x\to 2} \left(\frac{7x+2}{4-x}\right)^{2/3} = \left(\frac{7(2)+2}{4-2}\right)^{2/3} = 4.$

77. $\lim_{x\to 3} 10^{x^2-2x}$

SOLUTION $\lim_{x\to 3} 10^{x^2-2x} = 10^{3^2-2(3)} = 1000.$

78. $\lim_{x\to -\frac{\pi}{2}} 3^{\sin x}$

SOLUTION $\lim_{x\to-\frac{\pi}{2}} 3^{\sin x} = 3^{\sin(-\pi/2)} = \frac{1}{3}.$

79. $\lim_{x\to 4} \sin^{-1}\left(\frac{x}{4}\right)$

SOLUTION $\lim_{x\to 4} \sin^{-1}\left(\frac{x}{4}\right) = \sin^{-1}\left(\lim_{x\to 4}\frac{x}{4}\right) = \sin^{-1}\left(\frac{4}{4}\right) = \frac{\pi}{2}$

80. $\lim_{x\to 0} \tan^{-1}(e^x)$

SOLUTION $\lim_{x\to 0} \tan^{-1}(e^x) = \tan^{-1}\left(\lim_{x\to 0} e^x\right) = \tan^{-1}(e^0) = \tan^{-1} 1 = \frac{\pi}{4}$

81. Suppose that $f(x)$ and $g(x)$ are discontinuous at $x = c$. Does it follow that $f(x) + g(x)$ is discontinuous at $x = c$? If not, give a counterexample. Does this contradict Theorem 1 (i)?

SOLUTION Even if $f(x)$ and $g(x)$ are discontinuous at $x = c$, it is *not* necessarily true that $f(x) + g(x)$ is discontinuous at $x = c$. For example, suppose $f(x) = -x^{-1}$ and $g(x) = x^{-1}$. Both $f(x)$ and $g(x)$ are discontinuous at $x = 0$; however, the function $f(x) + g(x) = 0$, which is continuous everywhere, including $x = 0$. This does not contradict Theorem 1 (i), which deals only with continuous functions.

82. Prove that $f(x) = |x|$ is continuous for all x. *Hint:* To prove continuity at $x = 0$, consider the one-sided limits.

SOLUTION Let $c < 0$. Then

$$\lim_{x\to c} |x| = \lim_{x\to c} -x = -c = |c|.$$

Next, let $c > 0$. Then

$$\lim_{x\to c} |x| = \lim_{x\to c} x = c = |c|.$$

Finally,

$$\lim_{x\to 0-} |x| = \lim_{x\to 0-} -x = 0,$$

$$\lim_{x\to 0+} |x| = \lim_{x\to 0+} x = 0$$

and we recall that $|0| = 0$. Thus, $|x|$ is continuous for all x.

83. Use the result of Exercise 82 to prove that if $g(x)$ is continuous, then $f(x) = |g(x)|$ is also continuous.

SOLUTION Recall that the composition of two continuous functions is continuous. Now, $f(x) = |g(x)|$ is a composition of the continuous functions $g(x)$ and $|x|$, so is also continuous.

84. Which of the following quantities would be represented by continuous functions of time and which would have one or more discontinuities?

(a) Velocity of an airplane during a flight
(b) Temperature in a room under ordinary conditions
(c) Value of a bank account with interest paid yearly
(d) The salary of a teacher
(e) The population of the world

SOLUTION

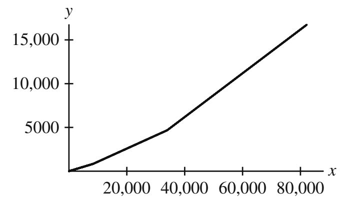
(a) The velocity of an airplane during a flight from Boston to Chicago is a continuous function of time.
(b) The temperature of a room under ordinary conditions is a continuous function of time.
(c) The value of a bank account with interest paid yearly is *not* a continuous function of time. It has discontinuities when deposits or withdrawals are made and when interest is paid.
(d) The salary of a teacher is *not* a continuous function of time. It has discontinuities whenever the teacher gets a raise (or whenever his or her salary is lowered).
(e) The population of the world is *not* a continuous function of time since it changes by a discrete amount with each birth or death. Since it takes on such large numbers (many billions), it is often treated as a continuous function for the purposes of mathematical modeling.

85. In 2009, the federal income tax $T(x)$ on income of x dollars (up to \$82,250) was determined by the formula

$$T(x) = \begin{cases} 0.10x & \text{for } 0 \le x < 8350 \\ 0.15x - 417.50 & \text{for } 8350 \le x < 33{,}950 \\ 0.25x - 3812.50 & \text{for } 33{,}950 \le x < 82{,}250 \end{cases}$$

Sketch the graph of $T(x)$. Does $T(x)$ have any discontinuities? Explain why, if $T(x)$ had a jump discontinuity, it might be advantageous in some situations to earn *less* money.

SOLUTION $T(x)$, the amount of federal income tax owed on an income of x dollars in 2009, might be a discontinuous function depending upon how the tax tables are constructed (as determined by that year's regulations). Here is a graph of $T(x)$ for that particular year.



If $T(x)$ had a jump discontinuity (say at $x = c$), it might be advantageous to earn slightly less income than c (say $c - \epsilon$) and be taxed at a lower rate than to earn c or more and be taxed at a higher rate. Your net earnings may actually be more in the former case than in the latter one.

Further Insights and Challenges

86. If $f(x)$ has a removable discontinuity at $x = c$, then it is possible to redefine $f(c)$ so that $f(x)$ is continuous at $x = c$. Can this be done in more than one way?

SOLUTION In order for $f(x)$ to have a removable discontinuity at $x = c$, $\lim_{x\to c} f(x) = L$ must exist. To remove the discontinuity, we define $f(c) = L$. Then f is continuous at $x = c$ since $\lim_{x\to c} f(x) = L = f(c)$. Now *assume* that we may define $f(c) = M \neq L$ and still have f continuous at $x = c$. Then $\lim_{x\to c} f(x) = f(c) = M$. Therefore $M = L$, a contradiction. Roughly speaking, there's only one way to fill in the hole in the graph of f!

87. Give an example of functions $f(x)$ and $g(x)$ such that $f(g(x))$ is continuous but $g(x)$ has at least one discontinuity.

SOLUTION Answers may vary. The simplest examples are the functions $f(g(x))$ where $f(x) = C$ is a constant function, and $g(x)$ is defined for all x. In these cases, $f(g(x)) = C$. For example, if $f(x) = 3$ and $g(x) = [x]$, g is discontinuous at all integer values $x = n$, but $f(g(x)) = 3$ is continuous.

88. Continuous at Only One Point Show that the following function is continuous only at $x = 0$:

$$f(x) = \begin{cases} x & \text{for } x \text{ rational} \\ -x & \text{for } x \text{ irrational} \end{cases}$$

SOLUTION Let $f(x) = x$ for x rational and $f(x) = -x$ for x irrational.

- Now $f(0) = 0$ since 0 is rational. Moreover, as $x \to 0$, we have $|f(x) - f(0)| = |f(x) - 0| = |x| \to 0$. Thus $\lim_{x\to 0} f(x) = f(0)$ and f is continuous at $x = 0$.
- Let $c \neq 0$ be any nonzero rational number. Let $\{x_1, x_2, \ldots\}$ be a sequence of irrational points that approach c; i.e., as $n \to \infty$, the x_n get arbitrarily close to c. Notice that as $n \to \infty$, we have $|f(x_n) - f(c)| = |-x_n - c| = |x_n + c| \to |2c| \neq 0$. Therefore, it is *not* true that $\lim_{x\to c} f(x) = f(c)$. Accordingly, f is *not* continuous at $x = c$. Since c was arbitrary, f is discontinuous at all rational numbers.
- Let $c \neq 0$ be any nonzero irrational number. Let $\{x_1, x_2, \ldots\}$ be a sequence of rational points that approach c; i.e., as $n \to \infty$, the x_n get arbitrarily close to c. Notice that as $n \to \infty$, we have $|f(x_n) - f(c)| = |x_n - (-c)| = |x_n + c| \to |2c| \neq 0$. Therefore, it is *not* true that $\lim_{x\to c} f(x) = f(c)$. Accordingly, f is *not* continuous at $x = c$. Since c was arbitrary, f is discontinuous at all irrational numbers.
- CONCLUSION: f is continuous at $x = 0$ and is discontinuous at all points $x \neq 0$.

89. Show that $f(x)$ is a discontinuous function for all x where $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 1 & \text{for } x \text{ rational} \\ -1 & \text{for } x \text{ irrational} \end{cases}$$

Show that $f(x)^2$ is continuous for all x.

SOLUTION $\lim_{x\to c} f(x)$ does not exist for any c. If c is irrational, then there is always a rational number r arbitrarily close to c so that $|f(c) - f(r)| = 2$. If, on the other hand, c is rational, there is always an *irrational* number z arbitrarily close to c so that $|f(c) - f(z)| = 2$.

On the other hand, $f(x)^2$ is a constant function that always has value 1, which is obviously continuous.

2.5 Evaluating Limits Algebraically

Preliminary Questions

1. Which of the following is indeterminate at $x = 1$?

$$\frac{x^2+1}{x-1}, \qquad \frac{x^2-1}{x+2}, \qquad \frac{x^2-1}{\sqrt{x+3}-2}, \qquad \frac{x^2+1}{\sqrt{x+3}-2}$$

SOLUTION At $x = 1$, $\frac{x^2-1}{\sqrt{x+3}-2}$ is of the form $\frac{0}{0}$; hence, this function is indeterminate. None of the remaining functions is indeterminate at $x = 1$: $\frac{x^2+1}{x-1}$ and $\frac{x^2+1}{\sqrt{x+3}-2}$ are undefined because the denominator is zero but the numerator is not, while $\frac{x^2-1}{x+2}$ is equal to 0.

2. Give counterexamples to show that these statements are false:

(a) If $f(c)$ is indeterminate, then the right- and left-hand limits as $x \to c$ are not equal.

(b) If $\lim_{x\to c} f(x)$ exists, then $f(c)$ is not indeterminate.

(c) If $f(x)$ is undefined at $x = c$, then $f(x)$ has an indeterminate form at $x = c$.

SOLUTION

(a) Let $f(x) = \frac{x^2-1}{x-1}$. At $x = 1$, f is indeterminate of the form $\frac{0}{0}$ but

$$\lim_{x\to 1-} \frac{x^2-1}{x-1} = \lim_{x\to 1-} (x+1) = 2 = \lim_{x\to 1+} (x+1) = \lim_{x\to 1+} \frac{x^2-1}{x-1}.$$

(b) Again, let $f(x) = \frac{x^2-1}{x-1}$. Then

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} (x+1) = 2$$

but $f(1)$ is indeterminate of the form $\frac{0}{0}$.

(c) Let $f(x) = \frac{1}{x}$. Then f is undefined at $x = 0$ but does not have an indeterminate form at $x = 0$.

3. The method for evaluating limits discussed in this section is sometimes called "simplify and plug in." Explain how it actually relies on the property of continuity.

SOLUTION If f is continuous at $x = c$, then, by definition, $\lim_{x\to c} f(x) = f(c)$; in other words, the limit of a continuous function at $x = c$ is the value of the function at $x = c$. The "simplify and plug-in" strategy is based on simplifying a function which is indeterminate to a continuous function. Once the simplification has been made, the limit of the remaining continuous function is obtained by evaluation.

Exercises

In Exercises 1–4, show that the limit leads to an indeterminate form. Then carry out the two-step procedure: Transform the function algebraically and evaluate using continuity.

1. $\lim_{x\to 6} \frac{x^2-36}{x-6}$

SOLUTION When we substitute $x = 6$ into $\frac{x^2-36}{x-6}$, we obtain the indeterminate form $\frac{0}{0}$. Upon factoring the numerator and simplifying, we find

$$\lim_{x\to 6} \frac{x^2-36}{x-6} = \lim_{x\to 6} \frac{(x-6)(x+6)}{x-6} = \lim_{x\to 6} (x+6) = 12.$$

2. $\lim_{h\to 3} \frac{9-h^2}{h-3}$

SOLUTION When we substitute $h = 3$ into $\frac{9-h^2}{h-3}$, we obtain the indeterminate form $\frac{0}{0}$. Upon factoring the denominator and simplifying, we find

$$\lim_{h\to 3} \frac{9-h^2}{h-3} = \lim_{h\to 3} \frac{(3-h)(3+h)}{h-3} = \lim_{h\to 3} -(3+h) = -6.$$

3. $\lim_{x \to -1} \frac{x^2+2x+1}{x+1}$

SOLUTION When we substitute $x = -1$ into $\frac{x^2+2x+1}{x+1}$, we obtain the indeterminate form $\frac{0}{0}$. Upon factoring the numerator and simplifying, we find

$$\lim_{x \to -1} \frac{x^2+2x+1}{x+1} = \lim_{x \to -1} \frac{(x+1)^2}{x+1} = \lim_{x \to -1} (x+1) = 0.$$

4. $\lim_{t \to 9} \frac{2t-18}{5t-45}$

SOLUTION When we substitute $t = 9$ into $\frac{2t-18}{5t-45}$, we obtain the indeterminate form $\frac{0}{0}$. Upon dividing out the common factor of $t-9$ from both the numerator and denominator, we find

$$\lim_{t \to 9} \frac{2t-18}{5t-45} = \lim_{t \to 9} \frac{2(t-9)}{5(t-9)} = \lim_{t \to 9} \frac{2}{5} = \frac{2}{5}.$$

In Exercises 5–34, evaluate the limit, if it exists. If not, determine whether the one-sided limits exist (finite or infinite).

5. $\lim_{x \to 7} \frac{x-7}{x^2-49}$

SOLUTION $\lim_{x \to 7} \frac{x-7}{x^2-49} = \lim_{x \to 7} \frac{x-7}{(x-7)(x+7)} = \lim_{x \to 7} \frac{1}{x+7} = \frac{1}{14}.$

6. $\lim_{x \to 8} \frac{x^2-64}{x-9}$

SOLUTION $\lim_{x \to 8} \frac{x^2-64}{x-9} = \frac{0}{-1} = 0$

7. $\lim_{x \to -2} \frac{x^2+3x+2}{x+2}$

SOLUTION $\lim_{x \to -2} \frac{x^2+3x+2}{x+2} = \lim_{x \to -2} \frac{(x+1)(x+2)}{x+2} = \lim_{x \to -2} (x+1) = -1.$

8. $\lim_{x \to 8} \frac{x^3-64x}{x-8}$

SOLUTION $\lim_{x \to 8} \frac{x^3-64x}{x-8} = \lim_{x \to 8} \frac{x(x-8)(x+8)}{x-8} = \lim_{x \to 8} x(x+8) = 8(16) = 128.$

9. $\lim_{x \to 5} \frac{2x^2-9x-5}{x^2-25}$

SOLUTION $\lim_{x \to 5} \frac{2x^2-9x-5}{x^2-25} = \lim_{x \to 5} \frac{(x-5)(2x+1)}{(x-5)(x+5)} = \lim_{x \to 5} \frac{2x+1}{x+5} = \frac{11}{10}.$

10. $\lim_{h \to 0} \frac{(1+h)^3-1}{h}$

SOLUTION

$$\lim_{h \to 0} \frac{(1+h)^3-1}{h} = \lim_{h \to 0} \frac{1+3h+3h^2+h^3-1}{h} = \lim_{h \to 0} \frac{3h+3h^2+h^3}{h}$$
$$= \lim_{h \to 0} (3+3h+h^2) = 3+3(0)+0^2 = 3.$$

11. $\lim_{x \to -\frac{1}{2}} \frac{2x+1}{2x^2+3x+1}$

SOLUTION $\lim_{x \to -\frac{1}{2}} \frac{2x+1}{2x^2+3x+1} = \lim_{x \to -\frac{1}{2}} \frac{2x+1}{(2x+1)(x+1)} = \lim_{x \to -\frac{1}{2}} \frac{1}{x+1} = 2.$

12. $\lim_{x \to 3} \frac{x^2-x}{x^2-9}$

SOLUTION As $x \to 3$, the numerator $x^2 - x \to 6$ while the denominator $x^2 - 9 \to 0$; thus, this limit does not exist. Checking the one-sided limits, we find

$$\lim_{x \to 3-} \frac{x^2-x}{x^2-9} = \lim_{x \to 3-} \frac{x(x-1)}{(x-3)(x+3)} = -\infty$$

while

$$\lim_{x \to 3+} \frac{x^2-x}{x^2-9} = \lim_{x \to 3+} \frac{x(x-1)}{(x-3)(x+3)} = \infty.$$

13. $\lim_{x\to 2} \frac{3x^2-4x-4}{2x^2-8}$

SOLUTION $\lim_{x\to 2} \frac{3x^2-4x-4}{2x^2-8} = \lim_{x\to 2} \frac{(3x+2)(x-2)}{2(x-2)(x+2)} = \lim_{x\to 2} \frac{3x+2}{2(x+2)} = \frac{8}{8} = 1.$

14. $\lim_{h\to 0} \frac{(3+h)^3-27}{h}$

SOLUTION

$$\lim_{h\to 0} \frac{(3+h)^3-27}{h} = \lim_{h\to 0} \frac{27+27h+9h^2+h^3-27}{h} = \lim_{h\to 0} \frac{27h+9h^2+h^3}{h}$$
$$= \lim_{h\to 0} (27+9h+h^2) = 27+9(0)+0^2 = 27.$$

15. $\lim_{t\to 0} \frac{4^{2t}-1}{4^t-1}$

SOLUTION $\lim_{t\ to 0} \frac{4^{2t}-1}{4^t-1} = \lim_{t\ to 0} \frac{(4^t-1)(4^t+1)}{4^t-1} = \lim_{t\to 0} (4^t+1) = 2.$

16. $\lim_{h\to 4} \frac{(h+2)^2-9h}{h-4}$

SOLUTION $\lim_{h\to 4} \frac{(h+2)^2-9h}{h-4} = \lim_{h\to 4} \frac{h^2-5h+4}{h-4} = \lim_{h\to 4} \frac{(h-1)(h-4)}{h-4} = \lim_{h\to 4} (h-1) = 3.$

17. $\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16}$

SOLUTION $\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16} = \lim_{x\to 16} \frac{\sqrt{x}-4}{\left(\sqrt{x}+4\right)\left(\sqrt{x}-4\right)} = \lim_{x\to 16} \frac{1}{\sqrt{x}+4} = \frac{1}{8}.$

18. $\lim_{t\to -2} \frac{2t+4}{12-3t^2}$

SOLUTION $\lim_{t\to -2} \frac{2t+4}{12-3t^2} = \lim_{t\to -2} \frac{2(t+2)}{-3(t-2)(t+2)} = \lim_{t\to -2} \frac{2}{-3(t-2)} = \frac{1}{6}.$

19. $\lim_{y\to 3} \frac{y^2+y-12}{y^3-10y+3}$

SOLUTION $\lim_{y\to 3} \frac{y^2+y-12}{y^3-10y+3} = \lim_{y\to 3} \frac{(y-3)(y+4)}{(y-3)(y^2+3y-1)} = \lim_{y\to 3} \frac{(y+4)}{(y^2+3y-1)} = \frac{7}{17}.$

20. $\lim_{h\to 0} \frac{\frac{1}{(h+2)^2}-\frac{1}{4}}{h}$

SOLUTION

$$\lim_{h\to 0} \frac{\frac{1}{(h+2)^2}-\frac{1}{4}}{h} = \lim_{h\to 0} \frac{\frac{4-(h+2)^2}{4(h+2)^2}}{h} = \lim_{h\to 0} \frac{\frac{4-(h^2+4h+4)}{4(h+2)^2}}{h} = \lim_{h\to 0} \frac{\frac{-h^2-4h}{4(h+2)^2}}{h}$$
$$= \lim_{h\to 0} \frac{h\frac{-h-4}{4(h+2)^2}}{h} = \lim_{h\to 0} \frac{-h-4}{4(h+2)^2} = \frac{-4}{16} = -\frac{1}{4}.$$

21. $\lim_{h\to 0} \frac{\sqrt{2+h}-2}{h}$

SOLUTION $\lim_{h\to 0} \frac{\sqrt{h+2}-2}{h}$ does not exist.

- As $h \to 0+$, we have $\frac{\sqrt{h+2}-2}{h} = \frac{\left(\sqrt{h+2}-2\right)\left(\sqrt{h+2}+2\right)}{h(\sqrt{h+2}+2)} = \frac{h-2}{h(\sqrt{h+2}+2)} \to -\infty.$
- As $h \to 0-$, we have $\frac{\sqrt{h+2}-2}{h} = \frac{\left(\sqrt{h+2}-2\right)\left(\sqrt{h+2}+2\right)}{h(\sqrt{h+2}+2)} = \frac{h-2}{h(\sqrt{h+2}+2)} \to \infty.$

22. $\lim_{x\to 8} \frac{\sqrt{x-4}-2}{x-8}$

SOLUTION

$$\lim_{x\to 8}\frac{\sqrt{x-4}-2}{x-8}=\lim_{x\to 8}\frac{(\sqrt{x-4}-2)(\sqrt{x-4}+2)}{(x-8)(\sqrt{x-4}+2)}=\lim_{x\to 8}\frac{x-4-4}{(x-8)(\sqrt{x-4}+2)}$$
$$=\lim_{x\to 8}\frac{1}{\sqrt{x-4}+2}=\frac{1}{\sqrt{4}+2}=\frac{1}{4}.$$

23. $\lim_{x\to 4}\dfrac{x-4}{\sqrt{x}-\sqrt{8-x}}$

SOLUTION

$$\lim_{x\to 4}\frac{x-4}{\sqrt{x}-\sqrt{8-x}}=\lim_{x\to 4}\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{(\sqrt{x}-\sqrt{8-x})(\sqrt{x}+\sqrt{8-x})}=\lim_{x\to 4}\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{x-(8-x)}$$
$$=\lim_{x\to 4}\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{2x-8}=\lim_{x\to 4}\frac{(x-4)(\sqrt{x}+\sqrt{8-x})}{2(x-4)}$$
$$=\lim_{x\to 4}\frac{(\sqrt{x}+\sqrt{8-x})}{2}=\frac{\sqrt{4}+\sqrt{4}}{2}=2.$$

24. $\lim_{x\to 4}\dfrac{\sqrt{5-x}-1}{2-\sqrt{x}}$

SOLUTION

$$\lim_{x\to 4}\frac{\sqrt{5-x}-1}{2-\sqrt{x}}=\lim_{x\to 4}\left(\frac{\sqrt{5-x}-1}{2-\sqrt{x}}\cdot\frac{\sqrt{5-x}+1}{\sqrt{5-x}+1}\right)=\lim_{x\to 4}\frac{4-x}{(2-\sqrt{x})(\sqrt{5-x}+1)}$$
$$=\lim_{x\to 4}\frac{(2-\sqrt{x})(2+\sqrt{x})}{(2-\sqrt{x})(\sqrt{5-x}+1)}=\lim_{x\to 4}\frac{2+\sqrt{x}}{\sqrt{5-x}+1}=2.$$

25. $\lim_{x\to 4}\left(\dfrac{1}{\sqrt{x}-2}-\dfrac{4}{x-4}\right)$

SOLUTION $\lim_{x\to 4}\left(\dfrac{1}{\sqrt{x}-2}-\dfrac{4}{x-4}\right)=\lim_{x\to 4}\dfrac{\sqrt{x}+2-4}{(\sqrt{x}-2)(\sqrt{x}+2)}=\lim_{x\to 4}\dfrac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)}=\dfrac{1}{4}.$

26. $\lim_{x\to 0+}\left(\dfrac{1}{\sqrt{x}}-\dfrac{1}{\sqrt{x^2+x}}\right)$

SOLUTION

$$\lim_{x\to 0+}\left(\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x^2+x}}\right)=\lim_{x\to 0+}\frac{\sqrt{x+1}-1}{\sqrt{x}\sqrt{x+1}}=\lim_{x\to 0+}\frac{\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)}{\sqrt{x}\sqrt{x+1}\left(\sqrt{x+1}+1\right)}$$
$$=\lim_{x\to 0+}\frac{x}{\sqrt{x}\sqrt{x+1}\left(\sqrt{x+1}+1\right)}=\lim_{x\to 0+}\frac{\sqrt{x}}{\sqrt{x+1}\left(\sqrt{x+1}+1\right)}=0.$$

27. $\lim_{x\to 0}\dfrac{\cot x}{\csc x}$

SOLUTION $\lim_{x\to 0}\dfrac{\cot x}{\csc x}=\lim_{x\to 0}\dfrac{\cos x}{\sin x}\cdot\sin x=\cos 0=1.$

28. $\lim_{\theta\to \frac{\pi}{2}}\dfrac{\cot\theta}{\csc\theta}$

SOLUTION $\lim_{\theta\to \frac{\pi}{2}}\dfrac{\cot\theta}{\csc\theta}=\lim_{\theta\to \frac{\pi}{2}}\dfrac{\cos\theta}{\sin\theta}\cdot\sin\theta=\cos\dfrac{\pi}{2}=0.$

29. $\lim_{t\to 2}\dfrac{2^{2t}+2^t-20}{2^t-4}$

SOLUTION $\lim_{t\to 2}\dfrac{2^{2t}+2^t-20}{2^t-4}=\lim_{t\to 2}\dfrac{(2^t+5)(2^t-4)}{2^t-4}=\lim_{t\to 2}(2^t+5)=9.$

30. $\lim_{x\to 1}\left(\dfrac{1}{1-x}-\dfrac{2}{1-x^2}\right)$

SOLUTION $\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{2}{1-x^2}\right)=\lim_{x\to 1}\frac{(1+x)-2}{(1-x)(1+x)}=\lim_{x\to 1}\frac{-1}{1+x}=-\frac{1}{2}.$

31. $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{\tan x-1}$

SOLUTION $\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{\tan x-1}\cdot\frac{\cos x}{\cos x}=\lim_{x\to\frac{\pi}{4}}\frac{(\sin x-\cos x)\cos x}{\sin x-\cos x}=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}.$

32. $\lim_{\theta\to\frac{\pi}{2}}\left(\sec\theta-\tan\theta\right)$

SOLUTION

$$\lim_{\theta\to\frac{\pi}{2}}\left(\sec\theta-\tan\theta\right)=\lim_{\theta\to\frac{\pi}{2}}\frac{1-\sin\theta}{\cos\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}=\lim_{\theta\to\frac{\pi}{2}}\frac{1-\sin^2\theta}{\cos\theta\,(1+\sin\theta)}=\lim_{\theta\to\frac{\pi}{2}}\frac{\cos\theta}{1+\sin\theta}=\frac{0}{2}=0.$$

33. $\lim_{\theta\to\frac{\pi}{4}}\left(\frac{1}{\tan\theta-1}-\frac{2}{\tan^2\theta-1}\right)$

SOLUTION $\lim_{\theta\to\frac{\pi}{4}}\left(\frac{1}{\tan\theta-1}-\frac{2}{\tan^2\theta-1}\right)=\lim_{\theta\to\frac{\pi}{4}}\frac{(\tan\theta+1)-2}{(\tan\theta+1)(\tan\theta-1)}=\lim_{\theta\to\frac{\pi}{4}}\frac{1}{\tan\theta+1}=\frac{1}{2}.$

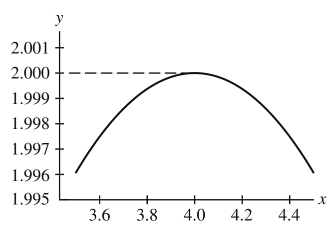
34. $\lim_{x\to\frac{\pi}{3}}\frac{2\cos^2 x+3\cos x-2}{2\cos x-1}$

SOLUTION

$$\lim_{x\to\frac{\pi}{3}}\frac{2\cos^2 x+3\cos x-2}{2\cos x-1}=\lim_{x\to\frac{\pi}{3}}\frac{(2\cos x-1)\,(\cos x+2)}{2\cos x-1}=\lim_{x\to\frac{\pi}{3}}\cos x+2=\cos\frac{\pi}{3}+2=\frac{5}{2}.$$

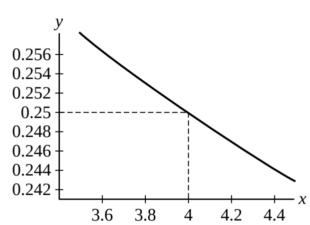
35. GU Use a plot of $f(x)=\frac{x-4}{\sqrt{x}-\sqrt{8-x}}$ to estimate $\lim_{x\to 4}f(x)$ to two decimal places. Compare with the answer obtained algebraically in Exercise 23.

SOLUTION Let $f(x)=\frac{x-4}{\sqrt{x}-\sqrt{8-x}}$. From the plot of $f(x)$ shown below, we estimate $\lim_{x\to 4}f(x)\approx 2.00$; to two decimal places, this matches the value of 2 obtained in Exercise 23.



36. GU Use a plot of $f(x)=\frac{1}{\sqrt{x}-2}-\frac{4}{x-4}$ to estimate $\lim_{x\to 4}f(x)$ numerically. Compare with the answer obtained algebraically in Exercise 25.

SOLUTION Let $f(x)=\frac{1}{\sqrt{x}-2}-\frac{4}{x-4}$. From the plot of $f(x)$ shown below, we estimate $\lim_{x\to 4}f(x)\approx 0.25$; to two decimal places, this matches the value of $\frac{1}{4}$ obtained in Exercise 25.



In Exercises 37–42, evaluate using the identity

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

37. $\displaystyle\lim_{x\to 2}\frac{x^3-8}{x-2}$

SOLUTION $\displaystyle\lim_{x\to 2}\frac{x^3-8}{x-2}=\lim_{x\to 2}\frac{(x-2)\left(x^2+2x+4\right)}{x-2}=\lim_{x\to 2}\left(x^2+2x+4\right)=12.$

38. $\displaystyle\lim_{x\to 3}\frac{x^3-27}{x^2-9}$

SOLUTION $\displaystyle\lim_{x\to 3}\frac{x^3-27}{x^2-9}=\lim_{x\to 3}\frac{(x-3)\left(x^2+3x+9\right)}{(x-3)(x+3)}=\lim_{x\to 3}\frac{\left(x^2+3x+9\right)}{x+3}=\frac{27}{6}=\frac{9}{2}.$

39. $\displaystyle\lim_{x\to 1}\frac{x^2-5x+4}{x^3-1}$

SOLUTION $\displaystyle\lim_{x\to 1}\frac{x^2-5x+4}{x^3-1}=\lim_{x\to 1}\frac{(x-1)(x-4)}{(x-1)\left(x^2+x+1\right)}=\lim_{x\to 1}\frac{x-4}{x^2+x+1}=-1.$

40. $\displaystyle\lim_{x\to -2}\frac{x^3+8}{x^2+6x+8}$

SOLUTION $\displaystyle\lim_{x\to -2}\frac{x^3+8}{x^2+6x+8}=\lim_{x\to -2}\frac{(x+2)(x^2-2x+4)}{(x+2)(x+4)}=\lim_{x\to -2}\frac{(x^2-2x+4)}{x+4}=\frac{12}{2}=6.$

41. $\displaystyle\lim_{x\to 1}\frac{x^4-1}{x^3-1}$

SOLUTION

$$\lim_{x\to 1}\frac{x^4-1}{x^3-1}=\lim_{x\to 1}\frac{(x^2-1)(x^2+1)}{(x-1)(x^2+x+1)}=\lim_{x\to 1}\frac{(x-1)(x+1)(x^2+1)}{(x-1)(x^2+x+1)}=\lim_{x\to 1}\frac{(x+1)(x^2+1)}{(x^2+x+1)}=\frac{4}{3}.$$

42. $\displaystyle\lim_{x\to 27}\frac{x-27}{x^{1/3}-3}$

SOLUTION $\displaystyle\lim_{x\to 27}\frac{x-27}{x^{1/3}-3}=\lim_{x\to 27}\frac{(x^{1/3}-3)(x^{2/3}+3x^{1/3}+9)}{x^{1/3}-3}=\lim_{x\to 27}(x^{2/3}+3x^{1/3}+9)=27$

43. Evaluate $\displaystyle\lim_{h\to 0}\frac{\sqrt[4]{1+h}-1}{h}$. *Hint:* Set $x=\sqrt[4]{1+h}$ and rewrite as a limit as $x\to 1$.

SOLUTION Let $x=\sqrt[4]{1+h}$. Then $h=x^4-1=(x-1)(x+1)(x^2+1)$, $x\to 1$ as $h\to 0$ and

$$\lim_{h\to 0}\frac{\sqrt[4]{1+h}-1}{h}=\lim_{x\to 1}\frac{x-1}{(x-1)(x+1)(x^2+1)}=\lim_{x\to 1}\frac{1}{(x+1)(x^2+1)}=\frac{1}{4}.$$

44. Evaluate $\displaystyle\lim_{h\to 0}\frac{\sqrt[3]{1+h}-1}{\sqrt[2]{1+h}-1}$. *Hint:* Set $x=\sqrt[6]{1+h}$ and rewrite as a limit as $x\to 1$.

SOLUTION Let $x=\sqrt[6]{1+h}$. Then $\sqrt[3]{1+h}-1=x^2-1=(x-1)(x+1)$, $\sqrt{1+h}-1=x^3-1=(x-1)(x^2+x+1)$, $x\to 1$ as $h\to 0$ and

$$\lim_{h\to 0}\frac{\sqrt[3]{1+h}-1}{\sqrt[2]{1+h}-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(x^2+x+1)}=\lim_{x\to 1}\frac{x+1}{x^2+x+1}=\frac{2}{3}.$$

In Exercises 45–54, evaluate in terms of the constant a.

45. $\displaystyle\lim_{x\to 0}(2a+x)$

SOLUTION $\displaystyle\lim_{x\to 0}(2a+x)=2a.$

46. $\displaystyle\lim_{h\to -2}(4ah+7a)$

SOLUTION $\displaystyle\lim_{h\to -2}(4ah+7a)=-a.$

47. $\displaystyle\lim_{t\to -1}(4t-2at+3a)$

SOLUTION $\displaystyle\lim_{t\to -1}(4t-2at+3a)=-4+5a.$

48. $\displaystyle\lim_{h\to 0}\frac{(3a+h)^2-9a^2}{h}$

SOLUTION $\lim_{h\to 0} \frac{(3a+h)^2 - 9a^2}{h} = \lim_{h\to 0} \frac{6ah + h^2}{h} = \lim_{h\to 0} (6a + h) = 6a.$

49. $\lim_{h\to 0} \frac{2(a+h)^2 - 2a^2}{h}$

SOLUTION $\lim_{h\to 0} \frac{2(a+h)^2 - 2a^2}{h} = \lim_{h\to 0} \frac{4ha + 2h^2}{h} = \lim_{h\to 0} (4a + 2h) = 4a.$

50. $\lim_{x\to a} \frac{(x+a)^2 - 4x^2}{x-a}$

SOLUTION

$$\lim_{x\to a} \frac{(x+a)^2 - 4x^2}{x-a} = \lim_{x\to a} \frac{(x^2 + 2ax + a^2) - 4x^2}{x-a} = \lim_{x\to a} \frac{-3x^2 + 2ax + a^2}{x-a}$$
$$= \lim_{x\to a} \frac{(a-x)(a+3x)}{x-a} = \lim_{x\to a} (-(a+3x)) = -4a.$$

51. $\lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{x-a}$

SOLUTION $\lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{x-a} = \lim_{x\to a} \frac{\sqrt{x} - \sqrt{a}}{\left(\sqrt{x} - \sqrt{a}\right)\left(\sqrt{x} + \sqrt{a}\right)} = \lim_{x\to a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{2\sqrt{a}}.$

52. $\lim_{h\to 0} \frac{\sqrt{a+2h} - \sqrt{a}}{h}$

SOLUTION

$$\lim_{h\to 0} \frac{\sqrt{a+2h} - \sqrt{a}}{h} = \lim_{h\to 0} \frac{\left(\sqrt{a+2h} - \sqrt{a}\right)\left(\sqrt{a+2h} + \sqrt{a}\right)}{h\left(\sqrt{a+2h} + \sqrt{a}\right)}$$
$$= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{a+2h} + \sqrt{a}\right)} = \lim_{h\to 0} \frac{2}{\sqrt{a+2h} + \sqrt{a}} = \frac{1}{\sqrt{a}}.$$

53. $\lim_{x\to 0} \frac{(x+a)^3 - a^3}{x}$

SOLUTION $\lim_{x\to 0} \frac{(x+a)^3 - a^3}{x} = \lim_{x\to 0} \frac{x^3 + 3x^2a + 3xa^2 + a^3 - a^3}{x} = \lim_{x\to 0} (x^2 + 3xa + 3a^2) = 3a^2.$

54. $\lim_{h\to a} \frac{\frac{1}{h} - \frac{1}{a}}{h-a}$

SOLUTION $\lim_{h\to a} \frac{\frac{1}{h} - \frac{1}{a}}{h-a} = \lim_{h\to a} \frac{\frac{a-h}{ah}}{h-a} = \lim_{h\to a} \frac{a-h}{ah} \frac{1}{h-a} = \lim_{h\to a} \frac{-1}{ah} = -\frac{1}{a^2}$

Further Insights and Challenges

In Exercises 55–58, find all values of c such that the limit exists.

55. $\lim_{x\to c} \frac{x^2 - 5x - 6}{x-c}$

SOLUTION $\lim_{x\to c} \frac{x^2 - 5x - 6}{x-c}$ will exist provided that $x - c$ is a factor of the numerator. (Otherwise there will be an infinite discontinuity at $x = c$.) Since $x^2 - 5x - 6 = (x+1)(x-6)$, this occurs for $c = -1$ and $c = 6$.

56. $\lim_{x\to 1} \frac{x^2 + 3x + c}{x-1}$

SOLUTION $\lim_{x\to 1} \frac{x^2 + 3x + c}{x-1}$ exists as long as $(x-1)$ is a factor of $x^2 + 3x + c$. If $x^2 + 3x + c = (x-1)(x+q)$, then $q - 1 = 3$ and $-q = c$. Hence $q = 4$ and $c = -4$.

57. $\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{c}{x^3-1}\right)$

SOLUTION Simplifying, we find

$$\frac{1}{x-1} - \frac{c}{x^3-1} = \frac{x^2 + x + 1 - c}{(x-1)(x^2 + x + 1)}.$$

In order for the limit to exist as $x \to 1$, the numerator must evaluate to 0 at $x = 1$. Thus, we must have $3 - c = 0$, which implies $c = 3$.

58. $\lim_{x\to 0} \dfrac{1 + cx^2 - \sqrt{1 + x^2}}{x^4}$

SOLUTION Rationalizing the numerator, we find

$$\frac{1 + cx^2 - \sqrt{1 + x^2}}{x^4} = \frac{(1 + cx^2 - \sqrt{1 + x^2})(1 + cx^2 + \sqrt{1 + x^2})}{x^4(1 + cx^2 + \sqrt{1 + x^2})} = \frac{(1 + cx^2)^2 - (1 + x^2)}{x^4(1 + cx^2 + \sqrt{1 + x^2})}$$
$$= \frac{(2c - 1)x^2 + c^2x^4}{x^4(1 + cx^2 + \sqrt{1 + x^2})}.$$

In order for the limit to exist as $x \to 0$, the coefficient of x^2 in the numerator must be zero. Thus, we need $2c - 1 = 0$, which implies $c = \frac{1}{2}$.

59. For which sign $\pm$ does the following limit exist?

$$\lim_{x\to 0}\left(\frac{1}{x} \pm \frac{1}{x(x-1)}\right)$$

SOLUTION

- The limit $\lim_{x\to 0}\left(\frac{1}{x} + \frac{1}{x(x-1)}\right) = \lim_{x\to 0} \frac{(x-1)+1}{x(x-1)} = \lim_{x\to 0} \frac{1}{x-1} = -1.$
- The limit $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{x(x-1)}\right)$ does not exist.
  - As $x \to 0+$, we have $\frac{1}{x} - \frac{1}{x(x-1)} = \frac{(x-1)-1}{x(x-1)} = \frac{x-2}{x(x-1)} \to \infty.$
  - As $x \to 0-$, we have $\frac{1}{x} - \frac{1}{x(x-1)} = \frac{(x-1)-1}{x(x-1)} = \frac{x-2}{x(x-1)} \to -\infty.$

2.6 Trigonometric Limits

Preliminary Questions

1. Assume that $-x^4 \le f(x) \le x^2$. What is $\lim_{x\to 0} f(x)$? Is there enough information to evaluate $\lim_{x\to \frac{1}{2}} f(x)$? Explain.

SOLUTION Since $\lim_{x\to 0} -x^4 = \lim_{x\to 0} x^2 = 0$, the squeeze theorem guarantees that $\lim_{x\to 0} f(x) = 0$. Since $\lim_{x\to \frac{1}{2}} -x^4 = -\frac{1}{16} \neq \frac{1}{4} = \lim_{x\to \frac{1}{2}} x^2$, we do not have enough information to determine $\lim_{x\to \frac{1}{2}} f(x)$.

2. State the Squeeze Theorem carefully.

SOLUTION Assume that for $x \neq c$ (in some open interval containing c),

$$l(x) \le f(x) \le u(x)$$

and that $\lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L$. Then $\lim_{x\to c} f(x)$ exists and

$$\lim_{x\to c} f(x) = L.$$

3. If you want to evaluate $\lim_{h\to 0} \dfrac{\sin 5h}{3h}$, it is a good idea to rewrite the limit in terms of the variable (choose one):

(a) $\theta = 5h$ **(b)** $\theta = 3h$ **(c)** $\theta = \dfrac{5h}{3}$

SOLUTION To match the given limit to the pattern of

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta},$$

it is best to substitute for the argument of the sine function; thus, rewrite the limit in terms of **(a):** $\theta = 5h$**.**

Exercises

1. State precisely the hypothesis and conclusions of the Squeeze Theorem for the situation in Figure 1.

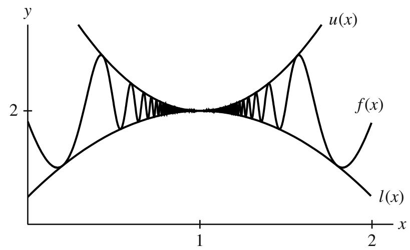


FIGURE 1

SOLUTION For all $x \neq 1$ on the open interval $(0, 2)$ containing $x = 1$, $\ell(x) \leq f(x) \leq u(x)$. Moreover,

$$\lim_{x\to 1} \ell(x) = \lim_{x\to 1} u(x) = 2.$$

Therefore, by the Squeeze Theorem,

$$\lim_{x\to 1} f(x) = 2.$$

2. In Figure 2, is $f(x)$ squeezed by $u(x)$ and $l(x)$ at $x = 3$? At $x = 2$?

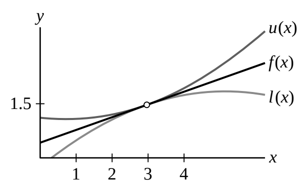


FIGURE 2

SOLUTION Because there is an open interval containing $x = 3$ on which $l(x) \leq f(x) \leq u(x)$ and $\lim_{x\to 3} l(x) = \lim_{x\to 3} u(x)$, $f(x)$ is *squeezed* by $u(x)$ and $l(x)$ at $x = 3$. Because there is an open interval containing $x = 2$ on which $l(x) \leq f(x) \leq u(x)$ but $\lim_{x\to 2} l(x) \neq \lim_{x\to 2} u(x)$, $f(x)$ is *trapped* by $u(x)$ and $l(x)$ at $x = 2$ but not *squeezed*.

3. What does the Squeeze Theorem say about $\lim_{x\to 7} f(x)$ if $\lim_{x\to 7} l(x) = \lim_{x\to 7} u(x) = 6$ and $f(x)$, $u(x)$, and $l(x)$ are related as in Figure 3? The inequality $f(x) \leq u(x)$ is not satisfied for all x. Does this affect the validity of your conclusion?

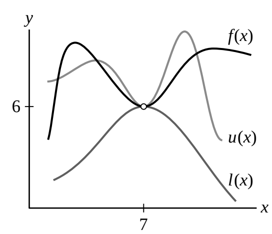


FIGURE 3

SOLUTION The Squeeze Theorem does not require that the inequalities $l(x) \leq f(x) \leq u(x)$ hold for all x, only that the inequalities hold on some open interval containing $x = c$. In Figure 3, it is clear that $l(x) \leq f(x) \leq u(x)$ on some open interval containing $x = 7$. Because $\lim_{x\to 7} u(x) = \lim_{x\to 7} l(x) = 6$, the Squeeze Theorem guarantees that $\lim_{x\to 7} f(x) = 6$.

4. Determine $\lim_{x\to 0} f(x)$ assuming that $\cos x \leq f(x) \leq 1$.

SOLUTION Because $\lim_{x\to 0} \cos x = \lim_{x\to 0} 1 = 1$, it follows that $\lim_{x\to 0} f(x) = 1$ by the Squeeze Theorem.

5. State whether the inequality provides sufficient information to determine $\lim_{x\to 1} f(x)$, and if so, find the limit.

(a) $4x - 5 \leq f(x) \leq x^2$

(b) $2x - 1 \leq f(x) \leq x^2$

(c) $4x - x^2 \leq f(x) \leq x^2 + 2$

SOLUTION

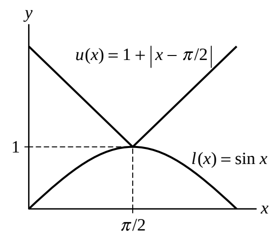
(a) Because $\lim_{x\to 1}(4x-5) = -1 \neq 1 = \lim_{x\to 1} x^2$, the given inequality does *not* provide sufficient information to determine $\lim_{x\to 1} f(x)$.

(b) Because $\lim_{x\to 1}(2x-1) = 1 = \lim_{x\to 1} x^2$, it follows from the Squeeze Theorem that $\lim_{x\to 1} f(x) = 1$.

(c) Because $\lim_{x\to 1}(4x-x^2) = 3 = \lim_{x\to 1}(x^2+2)$, it follows from the Squeeze Theorem that $\lim_{x\to 1} f(x) = 3$.

6. GU Plot the graphs of $u(x) = 1 + \left|x - \frac{\pi}{2}\right|$ and $l(x) = \sin x$ on the same set of axes. What can you say about $\lim_{x\to\frac{\pi}{2}} f(x)$ if $f(x)$ is squeezed by $l(x)$ and $u(x)$ at $x = \frac{\pi}{2}$?

SOLUTION



$\lim_{x\to\pi/2} u(x) = 1$ and $\lim_{x\to\pi/2} l(x) = 1$, so any function $f(x)$ satisfying $l(x) \le f(x) \le u(x)$ for all x near $\pi/2$ will satisfy $\lim_{x\to\pi/2} f(x) = 1$.

In Exercises 7–16, evaluate using the Squeeze Theorem.

7. $\lim_{x\to 0} x^2 \cos\frac{1}{x}$

SOLUTION Multiplying the inequality $-1 \le \cos\frac{1}{x} \le 1$, which holds for all $x \neq 0$, by x^2 yields $-x^2 \le x^2\cos\frac{1}{x} \le x^2$. Because

$$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{x\to 0} x^2\cos\frac{1}{x} = 0.$$

8. $\lim_{x\to 0} x\sin\frac{1}{x^2}$

SOLUTION Multiplying the inequality $\left|\sin\frac{1}{x^2}\right| \le 1$, which holds for $x \neq 0$, by $|x|$ yields $\left|x\sin\frac{1}{x^2}\right| \le |x|$ or $-|x| \le x\sin\frac{1}{x^2} \le |x|$. Because

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{x\to 0} x\sin\frac{1}{x^2} = 0.$$

9. $\lim_{x\to 1}(x-1)\sin\frac{\pi}{x-1}$

SOLUTION Multiplying the inequality $\left|\sin\frac{\pi}{x-1}\right| \le 1$, which holds for $x \neq 1$, by $|x-1|$ yields $\left|(x-1)\sin\frac{\pi}{x-1}\right| \le |x-1|$ or $-|x-1| \le (x-1)\sin\frac{\pi}{x-1} \le |x-1|$. Because

$$\lim_{x\to 1} -|x-1| = \lim_{x\to 1} |x-1| = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{x\to 1}(x-1)\sin\frac{\pi}{x-1} = 0.$$

10. $\lim_{x\to 3}(x^2-9)\frac{x-3}{|x-3|}$

SOLUTION For $x \neq 3$, $\frac{x-3}{|x-3|} = \pm 1$; thus

$$-|x^2 - 9| \leq (x^2 - 9)\frac{x-3}{|x-3|} \leq |x^2 - 9|.$$

Because

$$\lim_{x \to 3} -|x^2 - 9| = \lim_{x \to 3} |x^2 - 9| = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{x \to 3} (x^2 - 9)\frac{x-3}{|x-3|} = 0.$$

11. $\lim_{t \to 0} (2^t - 1) \cos \frac{1}{t}$

SOLUTION Multiplying the inequality $\left|\cos \frac{1}{t}\right| \leq 1$, which holds for $t \neq 0$, by $|2^t - 1|$ yields $\left|(2^t - 1) \cos \frac{1}{t}\right| \leq |2^t - 1|$ or $-|2^t - 1| \leq (2^t - 1) \cos \frac{1}{t} \leq |2^t - 1|$. Because

$$\lim_{t \to 0} -|2^t - 1| = \lim_{t \to 0} |2^t - 1| = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{t \to 0} (2^t - 1) \cos \frac{1}{t} = 0.$$

12. $\lim_{x \to 0+} \sqrt{x}\, e^{\cos(\pi/x)}$

SOLUTION Since $-1 \leq \cos \frac{\pi}{x} \leq 1$ and e^x is an increasing function, it follows that

$$\frac{1}{e} \leq e^{\cos(\pi/x)} \leq e \quad \text{and} \quad \frac{1}{e}\sqrt{x} \leq \sqrt{x} e^{\cos(\pi/x)} \leq e\sqrt{x}.$$

Because

$$\lim_{x \to 0+} \frac{1}{e}\sqrt{x} = \lim_{x \to 0+} e\sqrt{x} = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x \to 0+} \sqrt{x} e^{\cos(\pi/x)} = 0.$$

13. $\lim_{t \to 2} (t^2 - 4) \cos \frac{1}{t-2}$

SOLUTION Multiplying the inequality $\left|\cos \frac{1}{t-2}\right| \leq 1$, which holds for $t \neq 2$, by $|t^2 - 4|$ yields $\left|(t^2 - 4) \cos \frac{1}{t-2}\right| \leq |t^2 - 4|$ or $-|t^2 - 4| \leq (t^2 - 4) \cos \frac{1}{t-2} \leq |t^2 - 4|$. Because

$$\lim_{t \to 2} -|t^2 - 4| = \lim_{t \to 2} |t^2 - 4| = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{t \to 2} (t^2 - 4) \cos \frac{1}{t-2} = 0.$$

14. $\lim_{x \to 0} \tan x \cos\left(\sin \frac{1}{x}\right)$

SOLUTION Multiplying the inequality $\left|\cos\left(\sin \frac{1}{x}\right)\right| \leq 1$, which holds for $x \neq 0$, by $|\tan x|$ yields $\left|\tan x \cos\left(\sin \frac{1}{x}\right)\right| \leq |\tan x|$ or $-|\tan x| \leq \tan x \cos\left(\sin \frac{1}{x}\right) \leq |\tan x|$. Because

$$\lim_{x \to 0} -|\tan x| = \lim_{x \to 0} |\tan x| = 0,$$

it follows by the Squeeze Theorem that

$$\lim_{x \to 0} \tan x \cos\left(\sin \frac{1}{x}\right) = 0.$$

15. $\lim_{\theta \to \frac{\pi}{2}} \cos\theta \cos(\tan\theta)$

SOLUTION Multiplying the inequality $|\cos(\tan\theta)| \le 1$, which holds for all θ near $\frac{\pi}{2}$ but not equal to $\frac{\pi}{2}$, by $|\cos\theta|$ yields $|\cos\theta\cos(\tan\theta)| \le |\cos\theta|$ or $-|\cos\theta| \le \cos\theta\cos(\tan\theta) \le |\cos\theta|$. Because

$$\lim_{\theta \to \frac{\pi}{2}} -|\cos\theta| = \lim_{\theta \to \frac{\pi}{2}} |\cos\theta| = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{\theta \to \frac{\pi}{2}} \cos\theta\cos(\tan\theta) = 0.$$

16. $\lim_{t \to 0+} \sin t \tan^{-1}(\ln t)$

SOLUTION Multiplying the inequality $|\tan^{-1}(\ln t)| \le \frac{\pi}{2}$, which holds for all $t > 0$, by $|\sin t|$ yields $|\sin t \tan^{-1}(\ln t)| \le \frac{\pi}{2}|\sin t|$ or $-\frac{\pi}{2}|\sin t| \le \sin t \tan^{-1}(\ln t) \le \frac{\pi}{2}|\sin t|$. Because

$$\lim_{t \to 0+} -|\sin t| = \lim_{t \to 0+} |\sin t| = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{t \to 0+} \sin t \tan^{-1}(\ln t) = 0.$$

In Exercises 17–26, evaluate using Theorem 2 as necessary.

17. $\lim_{x \to 0} \frac{\tan x}{x}$

SOLUTION $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x}\frac{1}{\cos x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x} = 1 \cdot 1 = 1.$

18. $\lim_{x \to 0} \frac{\sin x \sec x}{x}$

SOLUTION $\lim_{x \to 0} \frac{\sin x \sec x}{x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \sec x = 1 \cdot 1 = 1.$

19. $\lim_{t \to 0} \frac{\sqrt{t^3+9}\sin t}{t}$

SOLUTION $\lim_{t \to 0} \frac{\sqrt{t^3+9}\sin t}{t} = \lim_{t \to 0} \sqrt{t^3+9} \cdot \lim_{t \to 0} \frac{\sin t}{t} = \sqrt{9} \cdot 1 = 3.$

20. $\lim_{t \to 0} \frac{\sin^2 t}{t}$

SOLUTION $\lim_{t \to 0} \frac{\sin^2 t}{t} = \lim_{t \to 0} \frac{\sin t}{t} \sin t = \lim_{t \to 0} \frac{\sin t}{t} \cdot \lim_{t \to 0} \sin t = 1 \cdot 0 = 0.$

21. $\lim_{x \to 0} \frac{x^2}{\sin^2 x}$

SOLUTION $\lim_{x \to 0} \frac{x^2}{\sin^2 x} = \lim_{x \to 0} \frac{1}{\frac{\sin x}{x}\frac{\sin x}{x}} = \lim_{x \to 0} \frac{1}{\frac{\sin x}{x}} \cdot \lim_{x \to 0} \frac{1}{\frac{\sin x}{x}} = \frac{1}{1} \cdot \frac{1}{1} = 1.$

22. $\lim_{t \to \frac{\pi}{2}} \frac{1-\cos t}{t}$

SOLUTION The function $\frac{1-\cos t}{t}$ is continuous at $\frac{\pi}{2}$; evaluate using substitution:

$$\lim_{t \to \frac{\pi}{2}} \frac{1-\cos t}{t} = \frac{1-0}{\frac{\pi}{2}} = \frac{2}{\pi}.$$

23. $\lim_{\theta \to 0} \frac{\sec\theta - 1}{\theta}$

SOLUTION $\lim_{\theta \to 0} \frac{\sec\theta - 1}{\theta} = \lim_{\theta \to 0} \frac{1-\cos\theta}{\theta\cos\theta} = \lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} \cdot \lim_{\theta \to 0} \frac{1}{\cos\theta} = 0 \cdot 1 = 0.$

24. $\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta}$

SOLUTION

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} \cdot \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 0 \cdot 1 = 0.$$

25. $\lim_{t\to \frac{\pi}{4}} \frac{\sin t}{t}$

SOLUTION $\frac{\sin t}{t}$ is continuous at $t = \frac{\pi}{4}$. Hence, by substitution

$$\lim_{t\to \frac{\pi}{4}} \frac{\sin t}{t} = \frac{\frac{\sqrt{2}}{2}}{\frac{\pi}{4}} = \frac{2\sqrt{2}}{\pi}.$$

26. $\lim_{t\to 0} \frac{\cos t - \cos^2 t}{t}$

SOLUTION By factoring and applying the Product Law:

$$\lim_{t\to 0} \frac{\cos t - \cos^2 t}{t} = \lim_{t\to 0} \cos t \cdot \lim_{t\to 0} \frac{1-\cos t}{t} = 1(0) = 0.$$

27. Let $L = \lim_{x\to 0} \frac{\sin 14x}{x}$.

(a) Show, by letting $\theta = 14x$, that $L = \lim_{\theta\to 0} 14\frac{\sin\theta}{\theta}$.

(b) Compute L.

SOLUTION

(a) Let $\theta = 14x$. Then $x = \frac{\theta}{14}$ and $\theta \to 0$ as $x \to 0$, so

$$L = \lim_{x\to 0} \frac{\sin 14x}{x} = \lim_{\theta\to 0} \frac{\sin\theta}{(\theta/14)} = \lim_{\theta\to 0} 14\frac{\sin\theta}{\theta}.$$

(b) Based on part (a),

$$L = 14 \lim_{\theta\to 0} \cdot \frac{\sin\theta}{\theta} = 14.$$

28. Evaluate $\lim_{h\to 0} \frac{\sin 9h}{\sin 7h}$. *Hint:* $\frac{\sin 9h}{\sin 7h} = \left(\frac{9}{7}\right)\left(\frac{\sin 9h}{9h}\right)\left(\frac{7h}{\sin 7h}\right)$.

SOLUTION

$$\lim_{h\to 0} \frac{\sin 9h}{\sin 7h} = \lim_{h\to 0} \frac{9}{7} \frac{(\sin 9h)\,/\,(9h)}{(\sin 7h)\,/\,(7h)} = \frac{9}{7} \frac{\lim_{h\to 0}(\sin 9h)/(9h)}{\lim_{h\to 0}(\sin 7h)/(7h)} = \frac{9}{7} \cdot \frac{1}{1} = \frac{9}{7}.$$

In Exercises 29–48, evaluate the limit.

29. $\lim_{h\to 0} \frac{\sin 9h}{h}$

SOLUTION $\lim_{h\to 0} \frac{\sin 9h}{h} = \lim_{h\to 0} 9\,\frac{\sin 9h}{9h} = 9.$

30. $\lim_{h\to 0} \frac{\sin 4h}{4h}$

SOLUTION Let $x = 4h$. Then $x \to 0$ as $h \to 0$ and

$$\lim_{h\to 0} \frac{\sin 4h}{4h} = \lim x \to 0 \frac{\sin x}{x} = 1.$$

31. $\lim_{h\to 0} \frac{\sin h}{5h}$

SOLUTION $\lim_{h\to 0} \frac{\sin h}{5h} = \lim_{h\to 0} \frac{1}{5} \frac{\sin h}{h} = \frac{1}{5}.$

32. $\lim_{x\to\frac{\pi}{6}} \dfrac{x}{\sin 3x}$

SOLUTION $\lim_{x\to\frac{\pi}{6}} \dfrac{x}{\sin 3x} = \dfrac{\pi/6}{\sin(\pi/2)} = \dfrac{\pi}{6}.$

33. $\lim_{\theta\to 0} \dfrac{\sin 7\theta}{\sin 3\theta}$

SOLUTION We have

$$\frac{\sin 7\theta}{\sin 3\theta} = \frac{7}{3}\left(\frac{\sin 7\theta}{7\theta}\right)\left(\frac{3\theta}{\sin 3\theta}\right)$$

Therefore,

$$\lim_{\theta\to 0}\frac{\sin 7\theta}{3\theta} = \frac{7}{3}\left(\lim_{\theta\to 0}\frac{\sin 7\theta}{7\theta}\right)\left(\lim_{\theta\to 0}\frac{3\theta}{\sin 3\theta}\right) = \frac{7}{3}(1)(1) = \frac{7}{3}$$

34. $\lim_{x\to 0} \dfrac{\tan 4x}{9x}$

SOLUTION $\lim_{x\to 0} \dfrac{\tan 4x}{9x} = \lim_{x\to 0} \dfrac{1}{9}\cdot\dfrac{\sin 4x}{4x}\cdot\dfrac{4}{\cos 4x} = \dfrac{4}{9}.$

35. $\lim_{x\to 0} x\csc 25x$

SOLUTION Let $h = 25x$. Then

$$\lim_{x\to 0} x\csc 25x = \lim_{h\to 0}\frac{h}{25}\csc h = \frac{1}{25}\lim_{h\to 0}\frac{h}{\sin h} = \frac{1}{25}.$$

36. $\lim_{t\to 0} \dfrac{\tan 4t}{t\sec t}$

SOLUTION $\lim_{t\to 0} \dfrac{\tan 4t}{t\sec t} = \lim_{t\to 0}\dfrac{4\sin 4t}{4t\,\cos(4t)\sec(t)} = \lim_{t\to 0}\dfrac{4\cos t}{\cos 4t}\cdot\dfrac{\sin 4t}{4t} = 4.$

37. $\lim_{h\to 0} \dfrac{\sin 2h\ \sin 3h}{h^2}$

SOLUTION

$$\lim_{h\to 0}\frac{\sin 2h\ \sin 3h}{h^2} = \lim_{h\to 0}\frac{\sin 2h\ \sin 3h}{h\cdot h} = \lim_{h\to 0}\frac{\sin 2h}{h}\,\frac{\sin 3h}{h}$$
$$= \lim_{h\to 0} 2\frac{\sin 2h}{2h}\,3\frac{\sin 3h}{3h} = \lim_{h\to 0} 2\frac{\sin 2h}{2h}\ \lim_{h\to 0} 3\frac{\sin 3h}{3h} = 2\cdot 3 = 6.$$

38. $\lim_{z\to 0} \dfrac{\sin(z/3)}{\sin z}$

SOLUTION $\lim_{z\to 0} \dfrac{\sin(z/3)}{\sin z}\cdot\dfrac{z/3}{z/3} = \lim_{z\to 0}\dfrac{1}{3}\cdot\dfrac{z}{\sin z}\cdot\dfrac{\sin(z/3)}{z/3} = \dfrac{1}{3}.$

39. $\lim_{\theta\to 0} \dfrac{\sin(-3\theta)}{\sin(4\theta)}$

SOLUTION $\lim_{\theta\to 0} \dfrac{\sin(-3\theta)}{\sin(4\theta)} = \lim_{\theta\to 0}\dfrac{-\sin(3\theta)}{3\theta}\cdot\dfrac{3}{4}\cdot\dfrac{4\theta}{\sin(4\theta)} = -\dfrac{3}{4}.$

40. $\lim_{x\to 0} \dfrac{\tan 4x}{\tan 9x}$

SOLUTION $\lim_{x\to 0} \dfrac{\tan 4x}{\tan 9x} = \lim_{x\to 0}\dfrac{\cos 9x}{\cos 4x}\cdot\dfrac{\sin 4x}{4x}\cdot\dfrac{4}{9}\cdot\dfrac{9x}{\sin 9x} = \dfrac{4}{9}.$

41. $\lim_{t\to 0} \dfrac{\csc 8t}{\csc 4t}$

SOLUTION $\lim_{t\to 0} \dfrac{\csc 8t}{\csc 4t} = \lim_{t\to 0}\dfrac{\sin 4t}{\sin 8t}\cdot\dfrac{8t}{4t}\cdot\dfrac{1}{2} = \dfrac{1}{2}.$

42. $\lim_{x\to 0} \dfrac{\sin 5x\sin 2x}{\sin 3x\sin 5x}$

SOLUTION $\lim_{x\to 0} \dfrac{\sin 5x\sin 2x}{\sin 3x\sin 5x} = \lim_{x\to 0}\dfrac{\sin 2x}{2x}\cdot\dfrac{2}{3}\cdot\dfrac{3x}{\sin 3x} = \dfrac{2}{3}.$

43. $\displaystyle\lim_{x\to 0} \frac{\sin 3x \sin 2x}{x \sin 5x}$

SOLUTION $\displaystyle\lim_{x\to 0} \frac{\sin 3x \sin 2x}{x \sin 5x} = \lim_{x\to 0} \left(3\,\frac{\sin 3x}{3x} \cdot \frac{2}{5}\,\frac{(\sin 2x)\,/\,(2x)}{(\sin 5x)\,/\,(5x)}\right) = \frac{6}{5}.$

44. $\displaystyle\lim_{h\to 0} \frac{1-\cos 2h}{h}$

SOLUTION $\displaystyle\lim_{h\to 0} \frac{1-\cos 2h}{h} = \lim_{h\to 0} 2\frac{1-\cos 2h}{2h} = 2\lim_{h\to 0} \frac{1-\cos 2h}{2h} = 2\cdot 0 = 0.$

45. $\displaystyle\lim_{h\to 0} \frac{\sin(2h)(1-\cos h)}{h^2}$

SOLUTION $\displaystyle\lim_{h\to 0} \frac{\sin(2h)(1-\cos h)}{h^2} = \lim_{h\to 0} \frac{\sin(2h)}{h} \lim_{h\to 0} \frac{1-\cos h}{h} = 1\cdot 0 = 0.$

46. $\displaystyle\lim_{t\to 0} \frac{1-\cos 2t}{\sin^2 3t}$

SOLUTION Using the identity $\cos 2t = 1 - 2\sin^2 t$, we find

$$\frac{1-\cos 2t}{\sin^2 3t} = \frac{2\sin^2 t}{\sin^2 3t} = \frac{2}{9}\left(\frac{\sin t}{t}\right)^2 \left(\frac{3t}{\sin 3t}\right)^2.$$

Thus,

$$\lim_{t\to 0} \frac{1-\cos 2t}{\sin^2 3t} = \lim_{t\to 0} \frac{2}{9}\left(\frac{\sin t}{t}\right)^2 \left(\frac{3t}{\sin 3t}\right)^2 = \frac{2}{9}.$$

47. $\displaystyle\lim_{\theta\to 0} \frac{\cos 2\theta - \cos\theta}{\theta}$

SOLUTION

$$\lim_{\theta\to 0} \frac{\cos 2\theta - \cos\theta}{\theta} = \lim_{\theta\to 0} \frac{(\cos 2\theta - 1) + (1-\cos\theta)}{\theta} = \lim_{\theta\to 0} \frac{\cos 2\theta - 1}{\theta} + \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta}$$
$$= -2\lim_{\theta\to 0} \frac{1-\cos 2\theta}{2\theta} + \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = -2\cdot 0 + 0 = 0.$$

48. $\displaystyle\lim_{h\to \frac{\pi}{2}} \frac{1-\cos 3h}{h}$

SOLUTION The function is continuous at $\frac{\pi}{2}$, so we may use substitution:

$$\lim_{h\to \frac{\pi}{2}} \frac{1-\cos 3h}{h} = \frac{1-\cos 3\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{1-0}{\frac{\pi}{2}} = \frac{2}{\pi}.$$

49. Calculate $\displaystyle\lim_{x\to 0-} \frac{\sin x}{|x|}$.

SOLUTION

$$\lim_{x\to 0-} \frac{\sin x}{|x|} = \lim_{x\to 0-} \frac{\sin x}{-x} = -1$$

50. Use the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ to evaluate the limit $\displaystyle\lim_{\theta\to 0} \frac{\sin 3\theta - 3\sin\theta}{\theta^3}$.

SOLUTION Using the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, we find

$$\frac{\sin 3\theta - 3\sin\theta}{\theta^3} = -4\left(\frac{\sin\theta}{\theta}\right)^3.$$

Therefore,

$$\lim_{\theta\to 0} \frac{\sin 3\theta - 3\sin\theta}{\theta^3} = -4\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^3 = -4(1)^3 = -4.$$

51. Prove the following result stated in Theorem 2:

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$$ 7

Hint: $\displaystyle\frac{1-\cos\theta}{\theta} = \frac{1}{1+\cos\theta}\cdot\frac{1-\cos^2\theta}{\theta}.$

SOLUTION

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\frac{1}{1+\cos\theta}\cdot\frac{1-\cos^2\theta}{\theta} = \lim_{\theta\to 0}\frac{1}{1+\cos\theta}\cdot\frac{\sin^2\theta}{\theta}$$
$$= \lim_{\theta\to 0}\frac{1}{1+\cos\theta}\cdot\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta} = \lim_{\theta\to 0}\frac{1}{1+\cos\theta}\cdot\lim_{\theta\to 0}\sin\theta\frac{\sin\theta}{\theta}$$
$$= \lim_{\theta\to 0}\frac{1}{1+\cos\theta}\cdot\lim_{\theta\to 0}\sin\theta\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \frac{1}{2}\cdot 0\cdot 1 = 0.$$

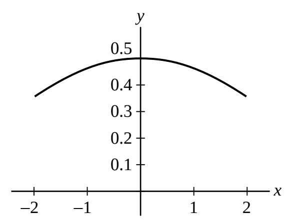
52. GU Investigate $\lim_{h\to 0}\frac{1-\cos h}{h^2}$ numerically (and graphically if you have a graphing utility). Then prove that the limit is equal to $\frac{1}{2}$. *Hint:* See the hint for Exercise 51.

SOLUTION

-

h	-0.1	-0.01	0.01	0.1
$\frac{1-\cos h}{h^2}$	0.499583	0.499996	0.499996	0.499583

The limit is $\frac{1}{2}$.

- 

- $\lim_{h\to 0}\frac{1-\cos h}{h^2} = \lim_{h\to 0}\frac{1-\cos^2 h}{h^2(1+\cos h)} = \lim_{h\to 0}\left(\frac{\sin h}{h}\right)^2\frac{1}{1+\cos h} = \frac{1}{2}.$

In Exercises 53–55, evaluate using the result of Exercise 52.

53. $\lim_{h\to 0}\frac{\cos 3h-1}{h^2}$

SOLUTION We make the substitution $\theta = 3h$. Then $h = \theta/3$, and

$$\lim_{h\to 0}\frac{\cos 3h-1}{h^2} = \lim_{\theta\to 0}\frac{\cos\theta-1}{(\theta/3)^2} = -9\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2} = -\frac{9}{2}.$$

54. $\lim_{h\to 0}\frac{\cos 3h-1}{\cos 2h-1}$

SOLUTION Write

$$\frac{\cos 3h-1}{\cos 2h-1} = \frac{1-\cos 3h}{(3h)^2}\cdot\frac{(2h)^2}{1-\cos 2h}\cdot\frac{9h^2}{4h^2}.$$

Then

$$\lim_{h\to 0}\frac{\cos 3h-1}{\cos 2h-1} = \frac{9}{4}\lim_{h\to 0}\frac{1-\cos 3h}{(3h)^2}\cdot\lim_{h\to 0}\frac{(2h)^2}{1-\cos 2h} = \frac{9}{4}\cdot\frac{1}{2}\cdot\frac{1}{1/2} = \frac{9}{4}.$$

55. $\lim_{t\to 0}\frac{\sqrt{1-\cos t}}{t}$

SOLUTION $\lim_{t\to 0+}\frac{\sqrt{1-\cos t}}{t} = \sqrt{\lim_{t\to 0+}\frac{1-\cos t}{t^2}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$; on the other hand, $\lim_{t\to 0-}\frac{\sqrt{1-\cos t}}{t} = -\sqrt{\lim_{t\to 0-}\frac{1-\cos t}{t^2}} = -\sqrt{\frac{1}{2}} = -\frac{\sqrt{2}}{2}$.

56. Use the Squeeze Theorem to prove that if $\lim_{x\to c} |f(x)| = 0$, then $\lim_{x\to c} f(x) = 0$.

SOLUTION Suppose $\lim_{x\to c} |f(x)| = 0$. Then

$$\lim_{x\to c} -|f(x)| = -\lim_{x\to c} |f(x)| = 0.$$

Now, for all x, the inequalities

$$-|f(x)| \le f(x) \le |f(x)|$$

hold. Because $\lim_{x\to c} |f(x)| = 0$ and $\lim_{x\to c} -|f(x)| = 0$, it follows from the Squeeze Theorem that $\lim_{x\to c} f(x) = 0$.

Further Insights and Challenges

57. Use the result of Exercise 52 to prove that for $m \neq 0$,

$$\lim_{x\to 0} \frac{\cos mx - 1}{x^2} = -\frac{m^2}{2}$$

SOLUTION Substitute $u = mx$ into $\dfrac{\cos mx - 1}{x^2}$. We obtain $x = \frac{u}{m}$. As $x \to 0$, $u \to 0$; therefore,

$$\lim_{x\to 0} \frac{\cos mx - 1}{x^2} = \lim_{u\to 0} \frac{\cos u - 1}{(u/m)^2} = \lim_{u\to 0} m^2 \frac{\cos u - 1}{u^2} = m^2\left(-\frac{1}{2}\right) = -\frac{m^2}{2}.$$

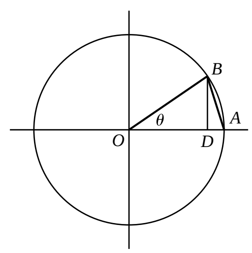
58. Using a diagram of the unit circle and the Pythagorean Theorem, show that

$$\sin^2\theta \le (1-\cos\theta)^2 + \sin^2\theta \le \theta^2$$

Conclude that $\sin^2\theta \le 2(1-\cos\theta) \le \theta^2$ and use this to give an alternative proof of Eq. (7) in Exercise 51. Then give an alternative proof of the result in Exercise 52.

SOLUTION

- Consider the unit circle shown below. The triangle BDA is a right triangle. It has base $1-\cos\theta$, altitude $\sin\theta$, and hypotenuse h. Observe that the hypotenuse h is less than the arc length AB = radius · angle $= 1\cdot\theta = \theta$. Apply the Pythagorean Theorem to obtain $(1-\cos\theta)^2 + \sin^2\theta = h^2 \le \theta^2$. The inequality $\sin^2\theta \le (1-\cos\theta)^2 + \sin^2\theta$ follows from the fact that $(1-\cos\theta)^2 \ge 0$.



- Note that

$$(1-\cos\theta)^2 + \sin^2\theta = 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta = 2 - 2\cos\theta = 2(1-\cos\theta).$$

  Therefore,

$$\sin^2\theta \le 2(1-\cos\theta) \le \theta^2.$$

- Divide the previous inequality by 2θ to obtain

$$\frac{\sin^2\theta}{2\theta} \le \frac{1-\cos\theta}{\theta} \le \frac{\theta}{2}.$$

  Because

$$\lim_{\theta\to 0} \frac{\sin^2\theta}{2\theta} = \frac{1}{2}\lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta\to 0} \sin\theta = \frac{1}{2}(1)(0) = 0,$$

  and $\lim_{h\to 0} \frac{\theta}{2} = 0$, it follows by the Squeeze Theorem that

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0.$$

- Divide the inequality

$$\sin^2\theta \le 2(1-\cos\theta) \le \theta^2$$

by $2\theta^2$ to obtain

$$\frac{\sin^2\theta}{2\theta^2} \le \frac{1-\cos\theta}{\theta^2} \le \frac{1}{2}.$$

Because

$$\lim_{\theta\to 0}\frac{\sin^2\theta}{2\theta^2} = \frac{1}{2}\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^2 = \frac{1}{2}(1^2) = \frac{1}{2},$$

and $\lim_{h\to 0}\frac{1}{2} = \frac{1}{2}$, it follows by the Squeeze Theorem that

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2} = \frac{1}{2}.$$

59. **(a)** Investigate $\lim_{x\to c}\frac{\sin x - \sin c}{x-c}$ numerically for the five values $c = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$.
(b) Can you guess the answer for general c?
(c) Check that your answer to (b) works for two other values of c.

SOLUTION
(a)

x	$c-0.01$	$c-0.001$	$c+0.001$	$c+0.01$
$\frac{\sin x-\sin c}{x-c}$	0.999983	0.99999983	0.99999983	0.999983

Here $c = 0$ and $\cos c = 1$.

x	$c-0.01$	$c-0.001$	$c+0.001$	$c+0.01$
$\frac{\sin x-\sin c}{x-c}$	0.868511	0.866275	0.865775	0.863511

Here $c = \frac{\pi}{6}$ and $\cos c = \frac{\sqrt{3}}{2} \approx 0.866025$.

x	$c-0.01$	$c-0.001$	$c+0.001$	$c+0.01$
$\frac{\sin x-\sin c}{x-c}$	0.504322	0.500433	0.499567	0.495662

Here $c = \frac{\pi}{3}$ and $\cos c = \frac{1}{2}$.

x	$c-0.01$	$c-0.001$	$c+0.001$	$c+0.01$
$\frac{\sin x-\sin c}{x-c}$	0.710631	0.707460	0.706753	0.703559

Here $c = \frac{\pi}{4}$ and $\cos c = \frac{\sqrt{2}}{2} \approx 0.707107$.

x	$c-0.01$	$c-0.001$	$c+0.001$	$c+0.01$
$\frac{\sin x-\sin c}{x-c}$	0.005000	0.000500	−0.000500	−0.005000

Here $c = \frac{\pi}{2}$ and $\cos c = 0$.

(b) $\lim_{x\to c}\frac{\sin x-\sin c}{x-c} = \cos c$.

(c)

x	$c-0.01$	$c-0.001$	$c+0.001$	$c+0.01$
$\frac{\sin x-\sin c}{x-c}$	−0.411593	−0.415692	−0.416601	−0.420686

Here $c = 2$ and $\cos c = \cos 2 \approx -0.416147$.

x	$c-0.01$	$c-0.001$	$c+0.001$	$c+0.01$
$\frac{\sin x - \sin c}{x-c}$	0.863511	0.865775	0.866275	0.868511

Here $c = -\frac{\pi}{6}$ and $\cos c = \frac{\sqrt{3}}{2} \approx 0.866025$.

2.7 Limits at Infinity

Preliminary Questions

1. Assume that

$$\lim_{x\to\infty} f(x) = L \quad \text{and} \quad \lim_{x\to L} g(x) = \infty$$

Which of the following statements are correct?

(a) $x = L$ is a vertical asymptote of $g(x)$.
(b) $y = L$ is a horizontal asymptote of $g(x)$.
(c) $x = L$ is a vertical asymptote of $f(x)$.
(d) $y = L$ is a horizontal asymptote of $f(x)$.

SOLUTION

(a) Because $\lim_{x\to L} g(x) = \infty$, $x = L$ is a vertical asymptote of $g(x)$. This statement is correct.
(b) This statement is not correct.
(c) This statement is not correct.
(d) Because $\lim_{x\to\infty} f(x) = L$, $y = L$ is a horizontal asymptote of $f(x)$. This statement is correct.

2. What are the following limits?

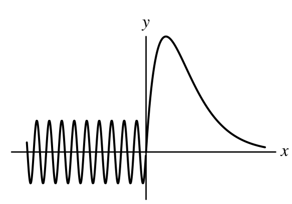
(a) $\lim_{x\to\infty} x^3$ **(b)** $\lim_{x\to-\infty} x^3$ **(c)** $\lim_{x\to-\infty} x^4$

SOLUTION

(a) $\lim_{x\to\infty} x^3 = \infty$
(b) $\lim_{x\to-\infty} x^3 = -\infty$
(c) $\lim_{x\to-\infty} x^4 = \infty$

3. Sketch the graph of a function that approaches a limit as $x \to \infty$ but does not approach a limit (either finite or infinite) as $x \to -\infty$.

SOLUTION



4. What is the sign of a if $f(x) = ax^3 + x + 1$ satisfies $\lim_{x\to-\infty} f(x) = \infty$?

SOLUTION Because $\lim_{x\to-\infty} x^3 = -\infty$, a must be negative to have $\lim_{x\to-\infty} f(x) = \infty$.

5. What is the sign of the leading coefficient a_7 if $f(x)$ is a polynomial of degree 7 such that $\lim_{x\to-\infty} f(x) = \infty$?

SOLUTION The behavior of $f(x)$ as $x \to -\infty$ is controlled by the leading term; that is, $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} a_7 x^7$. Because $x^7 \to -\infty$ as $x \to -\infty$, a_7 must be negative to have $\lim_{x\to-\infty} f(x) = \infty$.

6. Explain why $\lim_{x\to\infty} \sin\frac{1}{x}$ exists but $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist. What is $\lim_{x\to\infty} \sin\frac{1}{x}$?

SOLUTION As $x \to \infty$, $\frac{1}{x} \to 0$, so

$$\lim_{x\to\infty} \sin\frac{1}{x} = \sin 0 = 0.$$

On the other hand, $\frac{1}{x} \to \pm\infty$ as $x \to 0$, and as $\frac{1}{x} \to \pm\infty$, $\sin\frac{1}{x}$ oscillates infinitely often. Thus

$$\lim_{x\to 0} \sin\frac{1}{x}$$

does not exist.

Exercises

1. What are the horizontal asymptotes of the function in Figure 1?

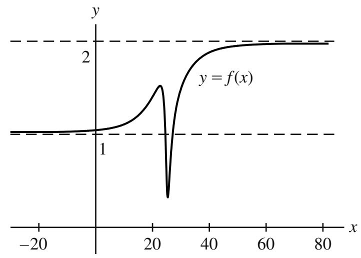


FIGURE 1

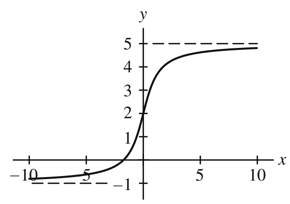
SOLUTION Because

$$\lim_{x\to-\infty} f(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} f(x) = 2,$$

the function $f(x)$ has horizontal asymptotes of $y = 1$ and $y = 2$.

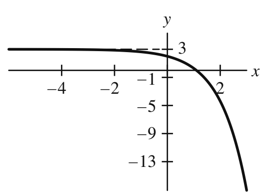
2. Sketch the graph of a function $f(x)$ that has both $y = -1$ and $y = 5$ as horizontal asymptotes.

SOLUTION



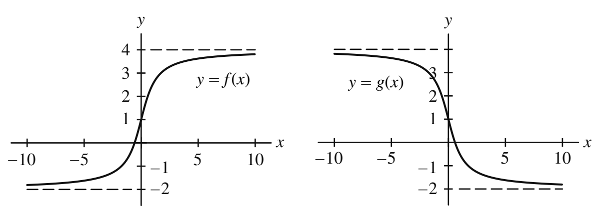
3. Sketch the graph of a function $f(x)$ with a single horizontal asymptote $y = 3$.

SOLUTION



4. Sketch the graphs of two functions $f(x)$ and $g(x)$ that have both $y = -2$ and $y = 4$ as horizontal asymptotes but $\lim_{x\to\infty} f(x) \neq \lim_{x\to\infty} g(x)$.

SOLUTION



5. GU Investigate the asymptotic behavior of $f(x) = \dfrac{x^3}{x^3 + x}$ numerically and graphically:

(a) Make a table of values of $f(x)$ for $x = \pm 50, \pm 100, \pm 500, \pm 1000$.

(b) Plot the graph of $f(x)$.

(c) What are the horizontal asymptotes of $f(x)$?

SOLUTION

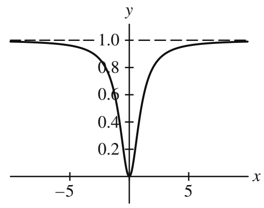
(a) From the table below, it appears that

$$\lim_{x\to\pm\infty} \frac{x^3}{x^3 + x} = 1.$$

x	± 50	± 100	± 500	± 1000
$f(x)$	0.999600	0.999900	0.999996	0.999999

(b) From the graph below, it also appears that

$$\lim_{x\to\pm\infty} \frac{x^3}{x^3+x} = 1.$$



(c) The horizontal asymptote of $f(x)$ is $y = 1$.

6. GU Investigate $\lim_{x\to\pm\infty} \frac{12x+1}{\sqrt{4x^2+9}}$ numerically and graphically:

(a) Make a table of values of $f(x) = \frac{12x+1}{\sqrt{4x^2+9}}$ for $x = \pm 100, \pm 500, \pm 1000, \pm 10{,}000$.

(b) Plot the graph of $f(x)$.

(c) What are the horizontal asymptotes of $f(x)$?

SOLUTION

(a) From the tables below, it appears that

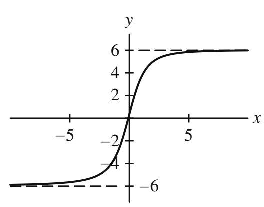
$$\lim_{x\to\infty} \frac{12x+1}{\sqrt{4x^2+9}} = 6 \quad \text{and} \quad \lim_{x\to-\infty} \frac{12x+1}{\sqrt{4x^2+9}} = -6.$$

x	-100	-500	-1000	-10000
$f(x)$	-5.994326	-5.998973	-5.999493	-5.999950

x	100	500	1000	10000
$f(x)$	6.004325	6.000973	6.000493	6.000050

(b) From the graph below, it also appears that

$$\lim_{x\to\infty} \frac{12x+1}{\sqrt{4x^2+9}} = 6 \quad \text{and} \quad \lim_{x\to-\infty} \frac{12x+1}{\sqrt{4x^2+9}} = -6.$$



(c) The horizontal asymptotes of $f(x)$ are $y = -6$ and $y = 6$.

In Exercises 7–16, evaluate the limit.

7. $\lim_{x\to\infty} \frac{x}{x+9}$

SOLUTION

$$\lim_{x\to\infty} \frac{x}{x+9} = \lim_{x\to\infty} \frac{x^{-1}(x)}{x^{-1}(x+9)} = \lim_{x\to\infty} \frac{1}{1+\frac{9}{x}} = \frac{1}{1+0} = 1.$$

8. $\lim_{x\to\infty} \frac{3x^2+20x}{4x^2+9}$

SOLUTION

$$\lim_{x\to\infty} \frac{3x^2+20x}{4x^2+9} = \lim_{x\to\infty} \frac{x^{-2}(3x^2+20x)}{x^{-2}(4x^2+9)} = \lim_{x\to\infty} \frac{3+\frac{20}{x}}{4+\frac{9}{x^2}} = \frac{3+0}{4+0} = \frac{3}{4}.$$

9. $\lim_{x\to\infty} \frac{3x^2+20x}{2x^4+3x^3-29}$

SOLUTION

$$\lim_{x\to\infty} \frac{3x^2+20x}{2x^4+3x^3-29} = \lim_{x\to\infty} \frac{x^{-4}(3x^2+20x)}{x^{-4}(2x^4+3x^3-29)} = \lim_{x\to\infty} \frac{\frac{3}{x^2}+\frac{20}{x^3}}{2+\frac{3}{x}-\frac{29}{x^4}} = \frac{0}{2} = 0.$$

10. $\lim_{x\to\infty} \frac{4}{x+5}$

SOLUTION

$$\lim_{x\to\infty} \frac{4}{x+5} = \lim_{x\to\infty} \frac{x^{-1}(4)}{x^{-1}(x+5)} = \lim_{x\to\infty} \frac{\frac{4}{x}}{1+\frac{5}{x}} = \frac{0}{1} = 0.$$

11. $\lim_{x\to\infty} \frac{7x-9}{4x+3}$

SOLUTION

$$\lim_{x\to\infty} \frac{7x-9}{4x+3} = \lim_{x\to\infty} \frac{x^{-1}(7x-9)}{x^{-1}(4x+3)} = \lim_{x\to\infty} \frac{7-\frac{9}{x}}{4+\frac{3}{x}} = \frac{7}{4}.$$

12. $\lim_{x\to\infty} \frac{9x^2-2}{6-29x}$

SOLUTION

$$\lim_{x\to\infty} \frac{9x^2-2}{6-29x} = \lim_{x\to\infty} \frac{x^{-1}(9x^2-2)}{x^{-1}(6-29x)} = \lim_{x\to\infty} \frac{9x-\frac{2}{x}}{\frac{6}{x}-29} = \frac{\infty}{-29} = -\infty.$$

13. $\lim_{x\to-\infty} \frac{7x^2-9}{4x+3}$

SOLUTION

$$\lim_{x\to-\infty} \frac{7x^2-9}{4x+3} = \lim_{x\to-\infty} \frac{x^{-1}(7x^2-9)}{x^{-1}(4x+3)} = \lim_{x\to-\infty} \frac{7x-\frac{9}{x}}{4+\frac{3}{x}} = -\infty.$$

14. $\lim_{x\to-\infty} \frac{5x-9}{4x^3+2x+7}$

SOLUTION

$$\lim_{x\to-\infty} \frac{5x-9}{4x^3+2x+7} = \lim_{x\to-\infty} \frac{x^{-3}(5x-9)}{x^{-3}(4x^3+2x+7)} = \lim_{x\to-\infty} \frac{\frac{5}{x^2}-\frac{9}{x^3}}{4+\frac{2}{x^2}+\frac{7}{x^3}} = \frac{0}{4} = 0.$$

15. $\lim_{x\to-\infty} \frac{3x^3-10}{x+4}$

SOLUTION

$$\lim_{x\to-\infty} \frac{3x^3-10}{x+4} = \lim_{x\to-\infty} \frac{x^{-1}(3x^3-10)}{x^{-1}(x+4)} = \lim_{x\to-\infty} \frac{3x^2-\frac{10}{x}}{1+\frac{4}{x}} = \frac{\infty}{1} = \infty.$$

16. $\lim_{x\to-\infty} \frac{2x^5+3x^4-31x}{8x^4-31x^2+12}$

SOLUTION

$$\lim_{x\to-\infty} \frac{2x^5+3x^4-31x}{8x^4-31x^2+12} = \lim_{x\to-\infty} \frac{x^{-4}(2x^5+3x^4-31x)}{x^{-4}(8x^4-31x^2+12)} = \lim_{x\to-\infty} \frac{2x+3-\frac{31}{x^3}}{8-\frac{31}{x^2}+\frac{12}{x^4}} = \frac{-\infty}{8} = -\infty.$$

In Exercises 17–22, find the horizontal asymptotes.

17. $f(x) = \frac{2x^2-3x}{8x^2+8}$

SOLUTION First calculate the limits as $x \to \pm\infty$. For $x \to \infty$,

$$\lim_{x\to\infty} \frac{2x^2 - 3x}{8x^2 + 8} = \lim_{x\to\infty} \frac{2 - \frac{3}{x}}{8 + \frac{8}{x^2}} = \frac{2}{8} = \frac{1}{4}.$$

Similarly,

$$\lim_{x\to-\infty} \frac{2x^2 - 3x}{8x^2 + 8} = \lim_{x\to-\infty} \frac{2 - \frac{3}{x}}{8 + \frac{8}{x^2}} = \frac{2}{8} = \frac{1}{4}.$$

Thus, the horizontal asymptote of $f(x)$ is $y = \frac{1}{4}$.

18. $f(x) = \dfrac{8x^3 - x^2}{7 + 11x - 4x^4}$

SOLUTION First calculate the limits as $x \to \pm\infty$. For $x \to \infty$,

$$\lim_{x\to\infty} \frac{8x^3 - x^2}{7 + 11x - 4x^4} = \lim_{x\to\infty} \frac{\frac{8}{x} - \frac{1}{x^2}}{\frac{7}{x^4} + \frac{11}{x^3} - 4} = 0.$$

Similarly,

$$\lim_{x\to-\infty} \frac{8x^3 - x^2}{7 + 11x - 4x^4} = \lim_{x\to-\infty} \frac{\frac{8}{x} - \frac{1}{x^2}}{\frac{7}{x^4} + \frac{11}{x^3} - 4} = 0.$$

Thus, the horizontal asymptote of $f(x)$ is $y = 0$.

19. $f(x) = \dfrac{\sqrt{36x^2 + 7}}{9x + 4}$

SOLUTION For $x > 0$, $x^{-1} = |x^{-1}| = \sqrt{x^{-2}}$, so

$$\lim_{x\to\infty} \frac{\sqrt{36x^2 + 7}}{9x + 4} = \lim_{x\to\infty} \frac{\sqrt{36 + \frac{7}{x^2}}}{9 + \frac{4}{x}} = \frac{\sqrt{36}}{9} = \frac{2}{3}.$$

On the other hand, for $x < 0$, $x^{-1} = -|x^{-1}| = -\sqrt{x^{-2}}$, so

$$\lim_{x\to-\infty} \frac{\sqrt{36x^2 + 7}}{9x + 4} = \lim_{x\to-\infty} \frac{-\sqrt{36 + \frac{7}{x^2}}}{9 + \frac{4}{x}} = \frac{-\sqrt{36}}{9} = -\frac{2}{3}.$$

Thus, the horizontal asymptotes of $f(x)$ are $y = \frac{2}{3}$ and $y = -\frac{2}{3}$.

20. $f(x) = \dfrac{\sqrt{36x^4 + 7}}{9x^2 + 4}$

SOLUTION For all $x \neq 0$, $x^{-2} = |x^{-2}| = \sqrt{x^{-4}}$, so

$$\lim_{x\to\infty} \frac{\sqrt{36x^4 + 7}}{9x^2 + 4} = \lim_{x\to\infty} \frac{\sqrt{36 + \frac{7}{x^4}}}{9 + \frac{4}{x^2}} = \frac{\sqrt{36}}{9} = \frac{2}{3}.$$

Similarly,

$$\lim_{x\to-\infty} \frac{\sqrt{36x^4 + 7}}{9x^2 + 4} = \lim_{x\to-\infty} \frac{\sqrt{36 + \frac{7}{x^4}}}{9 + \frac{4}{x^2}} = \frac{\sqrt{36}}{9} = \frac{2}{3}.$$

Thus, the horizontal asymptote of $f(x)$ is $y = \frac{2}{3}$.

21. $f(t) = \dfrac{e^t}{1 + e^{-t}}$

SOLUTION With

$$\lim_{t\to\infty} \frac{e^t}{1 + e^{-t}} = \frac{\infty}{1} = \infty$$

and

$$\lim_{t\to-\infty} \frac{e^t}{1 + e^{-t}} = 0,$$

the function $f(t)$ has one horizontal asymptote, $y = 0$.

22. $f(t) = \dfrac{t^{1/3}}{(64t^2+9)^{1/6}}$

SOLUTION For $t > 0$, $t^{-1/3} = |t^{-1/3}| = (t^{-2})^{1/6}$, so

$$\lim_{t\to\infty} \frac{t^{1/3}}{(64t^2+9)^{1/6}} = \lim_{t\to\infty} \frac{1}{(64+\frac{9}{t^2})^{1/6}} = \frac{1}{2}.$$

On the other hand, for $t < 0$, $t^{-1/3} = -|t^{-1/3}| = -(t^{-2})^{1/6}$, so

$$\lim_{t\to-\infty} \frac{t^{1/3}}{(64t^2+9)^{1/6}} = \lim_{t\to-\infty} \frac{1}{-(64+\frac{9}{t^2})^{1/6}} = -\frac{1}{2}.$$

Thus, the horizontal asymptotes for $f(t)$ are $y = \frac{1}{2}$ and $y = -\frac{1}{2}$.

In Exercises 23–30, evaluate the limit.

23. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{9x^4+3x+2}}{4x^3+1}$

SOLUTION For $x > 0$, $x^{-3} = |x^{-3}| = \sqrt{x^{-6}}$, so

$$\lim_{x\to\infty} \frac{\sqrt{9x^4+3x+2}}{4x^3+1} = \lim_{x\to\infty} \frac{\sqrt{\frac{9}{x^2}+\frac{3}{x^5}+\frac{2}{x^6}}}{4+\frac{1}{x^3}} = 0.$$

24. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^3+20x}}{10x-2}$

SOLUTION For $x > 0$, $x^{-1} = |x^{-1}| = \sqrt{x^{-2}}$, so

$$\lim_{x\to\infty} \frac{\sqrt{x^3+20x}}{10x-2} = \lim_{x\to\infty} \frac{\sqrt{x+\frac{20}{x}}}{10-\frac{2}{x}} = \frac{\infty}{10} = \infty.$$

25. $\displaystyle\lim_{x\to-\infty} \frac{8x^2+7x^{1/3}}{\sqrt{16x^4+6}}$

SOLUTION For $x < 0$, $x^{-2} = |x^{-2}| = \sqrt{x^{-4}}$, so

$$\lim_{x\to-\infty} \frac{8x^2+7x^{1/3}}{\sqrt{16x^4+6}} = \lim_{x\to-\infty} \frac{8+\frac{7}{x^{5/3}}}{\sqrt{16+\frac{6}{x^4}}} = \frac{8}{\sqrt{16}} = 2.$$

26. $\displaystyle\lim_{x\to-\infty} \frac{4x-3}{\sqrt{25x^2+4x}}$

SOLUTION For $x < 0$, $x^{-1} = -|x^{-1}| = -\sqrt{x^{-2}}$, so

$$\lim_{x\to-\infty} \frac{4x-3}{\sqrt{25x^2+4x}} = \lim_{x\to-\infty} \frac{4-\frac{3}{x}}{-\sqrt{25+\frac{4}{x}}} = \frac{4}{-\sqrt{25}} = -\frac{4}{5}.$$

27. $\displaystyle\lim_{t\to\infty} \frac{t^{4/3}+t^{1/3}}{(4t^{2/3}+1)^2}$

SOLUTION $\displaystyle\lim_{t\to\infty} \frac{t^{4/3}+t^{1/3}}{(4t^{2/3}+1)^2} = \lim_{t\to\infty} \frac{1+\frac{1}{t}}{(4+\frac{1}{t^{2/3}})^2} = \frac{1}{16}.$

28. $\displaystyle\lim_{t\to\infty} \frac{t^{4/3}-9t^{1/3}}{(8t^4+2)^{1/3}}$

SOLUTION $\displaystyle\lim_{t\to\infty} \frac{t^{4/3}-9t^{1/3}}{(8t^4+2)^{1/3}} = \lim_{t\to\infty} \frac{1-\frac{9}{t}}{(8+\frac{2}{t^4})^{1/3}} = \frac{1}{2}.$

29. $\displaystyle\lim_{x\to-\infty} \frac{|x|+x}{x+1}$

SOLUTION For $x < 0$, $|x| = -x$. Therefore, for all $x < 0$,

$$\frac{|x|+x}{x+1} = \frac{-x+x}{x+1} = 0;$$

consequently,

$$\lim_{x\to-\infty} \frac{|x|+x}{x+1} = 0.$$

30. $\displaystyle\lim_{t\to-\infty}\frac{4+6e^{2t}}{5-9e^{3t}}$

SOLUTION Because

$$\lim_{t\to-\infty}e^{2t}=\lim_{t\to-\infty}e^{3t}=0,$$

it follows that

$$\lim_{t\to-\infty}\frac{4+6e^{2t}}{5-9e^{3t}}=\frac{4+0}{5-0}=\frac{4}{5}.$$

31. Determine $\displaystyle\lim_{x\to\infty}\tan^{-1}x$. Explain geometrically.

SOLUTION As an angle θ increases from 0 to $\frac{\pi}{2}$, its tangent $x=\tan\theta$ approaches ∞. Therefore,

$$\lim_{x\to\infty}\tan^{-1}x=\frac{\pi}{2}.$$

Geometrically, this means that the graph of $y=\tan^{-1}x$ has a horizontal asymptote at $y=\frac{\pi}{2}$.

32. Show that $\displaystyle\lim_{x\to\infty}(\sqrt{x^2+1}-x)=0$. *Hint:* Observe that

$$\sqrt{x^2+1}-x=\frac{1}{\sqrt{x^2+1}+x}$$

SOLUTION Rationalizing the "numerator," we find

$$\begin{aligned}\sqrt{x^2+1}-x&=(\sqrt{x^2+1}-x)\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\\&=\frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x}=\frac{1}{\sqrt{x^2+1}+x}.\end{aligned}$$

Thus,

$$\lim_{x\to\infty}(\sqrt{x^2+1}-x)=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}=0.$$

33. According to the **Michaelis–Menten equation** (Figure 7), when an enzyme is combined with a substrate of concentration s (in millimolars), the reaction rate (in micromolars/min) is

$$R(s)=\frac{As}{K+s}\qquad(A,\ K\ \text{constants})$$

(a) Show, by computing $\displaystyle\lim_{s\to\infty}R(s)$, that A is the limiting reaction rate as the concentration s approaches ∞.

(b) Show that the reaction rate $R(s)$ attains one-half of the limiting value A when $s=K$.

(c) For a certain reaction, $K=1.25$ mM and $A=0.1$. For which concentration s is $R(s)$ equal to 75% of its limiting value?

Leonor Michaelis 1875–1949 Maud Menten 1879–1960

FIGURE 2 Canadian-born biochemist Maud Menten is best known for her fundamental work on enzyme kinetics with German scientist Leonor Michaelis. She was also an accomplished painter, clarinetist, mountain climber, and master of numerous languages.

SOLUTION

(a) $\displaystyle\lim_{s\to\infty}R(s)=\lim_{s\to\infty}\frac{As}{K+s}=\lim_{s\to\infty}\frac{A}{1+\frac{K}{s}}=A.$

(b) Observe that

$$R(K) = \frac{AK}{K+K} = \frac{AK}{2K} = \frac{A}{2},$$

have of the limiting value.

(c) By part (a), the limiting value is 0.1, so we need to determine the value of s that satisfies

$$R(s) = \frac{0.1s}{1.25+s} = 0.075.$$

Solving this equation for s yields

$$s = \frac{(1.25)(0.075)}{0.025} = 3.75 \text{ mM}.$$

34. Suppose that the average temperature of the earth is $T(t) = 283 + 3(1 - e^{-0.03t})$ kelvins, where t is the number of years since 2000.

(a) Calculate the long-term average $L = \lim_{t\to\infty} T(t)$.

(b) At what time is $T(t)$ within one-half a degree of its limiting value?

SOLUTION

(a) $L = \lim_{t\to\infty} T(t) = \lim_{t\to\infty} (283 + 3(1 - e^{-0.03t})) = 286$ kelvins.

(b) We need to solve the equation

$$T(t) = 283 + 3(1 - e^{-0.03t}) = 285.5.$$

This yields

$$t = \frac{1}{0.03}\ln 6 \approx 59.73.$$

The average temperature of the earth will be within one-half a degree of its limiting value in roughly 2060.

In Exercises 35–42, calculate the limit.

35. $\lim_{x\to\infty} \left(\sqrt{4x^4+9x} - 2x^2\right)$

SOLUTION Write

$$\sqrt{4x^4+9x} - 2x^2 = \left(\sqrt{4x^4+9x} - 2x^2\right)\frac{\sqrt{4x^4+9x}+2x^2}{\sqrt{4x^4+9x}+2x^2}$$

$$= \frac{(4x^4+9x) - 4x^4}{\sqrt{4x^4+9x}+2x^2} = \frac{9x}{\sqrt{4x^4+9x}+2x^2}.$$

Thus,

$$\lim_{x\to\infty} (\sqrt{4x^4+9x} - 2x^2) = \lim_{x\to\infty} \frac{9x}{\sqrt{4x^4+9x}+2x^2} = 0.$$

36. $\lim_{x\to\infty} (\sqrt{9x^3+x} - x^{3/2})$

SOLUTION Write

$$\sqrt{9x^3+x} - x^{3/2} = \left(\sqrt{9x^3+x} - x^{3/2}\right)\frac{\sqrt{9x^3+x}+x^{3/2}}{\sqrt{9x^3+x}+x^{3/2}}$$

$$= \frac{(9x^3+x) - x^3}{\sqrt{9x^3+x}+x^{3/2}} = \frac{8x^3+x}{\sqrt{9x^3+x}+x^{3/2}}.$$

Thus,

$$\lim_{x\to\infty} (\sqrt{9x^3+x} - x^{3/2}) = \lim_{t\to\infty} \frac{8x^3+x}{\sqrt{9x^3+x}+x^{3/2}} = \infty.$$

37. $\lim_{x\to\infty} \left(2\sqrt{x} - \sqrt{x+2}\right)$

SOLUTION Write

$$2\sqrt{x} - \sqrt{x+2} = \left(2\sqrt{x} - \sqrt{x+2}\right)\frac{2\sqrt{x} + \sqrt{x+2}}{2\sqrt{x} + \sqrt{x+2}}$$

$$= \frac{4x - (x+2)}{2\sqrt{x} + \sqrt{x+2}} = \frac{3x-2}{2\sqrt{x} + \sqrt{x+2}}.$$

Thus,

$$\lim_{x\to\infty}(2\sqrt{x} - \sqrt{x+2}) = \lim_{x\to\infty}\frac{3x-2}{2\sqrt{x} + \sqrt{x+2}} = \infty.$$

38. $\lim_{x\to\infty}\left(\frac{1}{x} - \frac{1}{x+2}\right)$

SOLUTION $\lim_{x\to\infty}\left(\frac{1}{x} - \frac{1}{x+2}\right) = \lim_{x\to\infty}\frac{2}{x(x+2)} = 0.$

39. $\lim_{x\to\infty}(\ln(3x+1) - \ln(2x+1))$

SOLUTION Because

$$\ln(3x+1) - \ln(2x+1) = \ln\frac{3x+1}{2x+1}$$

and

$$\lim_{x\to\infty}\frac{3x+1}{2x+1} = \frac{3}{2},$$

it follows that

$$\lim_{x\to\infty}(\ln(3x+1) - \ln(2x+1)) = \ln\frac{3}{2}.$$

40. $\lim_{x\to\infty}\left(\ln(\sqrt{5x^2+2}) - \ln x\right)$

SOLUTION Because

$$\ln(\sqrt{5x^2+2}) - \ln x = \ln\frac{\sqrt{5x^2+2}}{x}$$

and

$$\lim_{x\to\infty}\frac{\sqrt{5x^2+2}}{x} = \lim_{x\to\infty}\frac{\sqrt{5+\frac{2}{x^2}}}{1} = \sqrt{5},$$

it follows that

$$\lim_{x\to\infty}\left(\ln(\sqrt{5x^2+2}) - \ln x\right) = \ln\sqrt{5} = \frac{1}{2}\ln 5.$$

41. $\lim_{x\to\infty}\tan^{-1}\left(\frac{x^2+9}{9-x}\right)$

SOLUTION Because

$$\lim_{x\to\infty}\frac{x^2+9}{9-x} = \lim_{x\to\infty}\frac{x+\frac{9}{x}}{\frac{9}{x}-1} = \frac{\infty}{-1} = -\infty,$$

it follows that

$$\lim_{x\to\infty}\tan^{-1}\left(\frac{x^2+9}{9-x}\right) = -\frac{\pi}{2}.$$

42. $\lim_{x\to\infty}\tan^{-1}\left(\frac{1+x}{1-x}\right)$

SOLUTION Because

$$\lim_{x\to\infty}\frac{1+x}{1-x} = -1,$$

it follows that

$$\lim_{x\to\infty}\tan^{-1}\left(\frac{1+x}{1-x}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}.$$

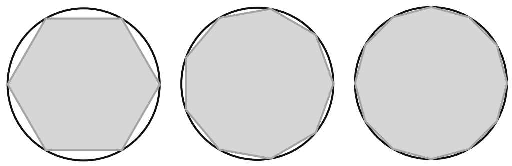
43. Let $P(n)$ be the perimeter of an n-gon inscribed in a unit circle (Figure 3).

(a) Explain, intuitively, why $P(n)$ approaches 2π as $n \to \infty$.

(b) Show that $P(n) = 2n \sin\left(\frac{\pi}{n}\right)$.

(c) Combine (a) and (b) to conclude that $\lim_{n\to\infty} \frac{n}{\pi} \sin\left(\frac{\pi}{n}\right) = 1$.

(d) Use this to give another argument that $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$.



$n = 6$ $n = 9$ $n = 12$

FIGURE 3

SOLUTION

(a) As $n \to \infty$, the n-gon approaches a circle of radius 1. Therefore, the perimeter of the n-gon approaches the circumference of the unit circle as $n \to \infty$. That is, $P(n) \to 2\pi$ as $n \to \infty$.

(b) Each side of the n-gon is the third side of an isosceles triangle with equal length sides of length 1 and angle $\theta = \frac{2\pi}{n}$ between the equal length sides. The length of each side of the n-gon is therefore

$$\sqrt{1^2 + 1^2 - 2\cos\frac{2\pi}{n}} = \sqrt{2(1 - \cos\frac{2\pi}{n})} = \sqrt{4\sin^2\frac{\pi}{n}} = 2\sin\frac{\pi}{n}.$$

Finally,

$$P(n) = 2n \sin\frac{\pi}{n}.$$

(c) Combining parts (a) and (b),

$$\lim_{n\to\infty} P(n) = \lim_{n\to\infty} 2n \sin\frac{\pi}{n} = 2\pi.$$

Dividing both sides of this last expression by 2π yields

$$\lim_{n\to\infty} \frac{n}{\pi} \sin\frac{\pi}{n} = 1.$$

(d) Let $\theta = \frac{\pi}{n}$. Then $\theta \to 0$ as $n \to \infty$,

$$\frac{n}{\pi} \sin\frac{\pi}{n} = \frac{1}{\theta}\sin\theta = \frac{\sin\theta}{\theta},$$

and

$$\lim_{n\to\infty} \frac{n}{\pi} \sin\frac{\pi}{n} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

44. Physicists have observed that Einstein's theory of **special relativity** reduces to Newtonian mechanics in the limit as $c \to \infty$, where c is the speed of light. This is illustrated by a stone tossed up vertically from ground level so that it returns to earth one second later. Using Newton's Laws, we find that the stone's maximum height is $h = g/8$ meters ($g = 9.8$ m/s^2). According to special relativity, the stone's mass depends on its velocity divided by c, and the maximum height is

$$h(c) = c\sqrt{c^2/g^2 + 1/4} - c^2/g$$

Prove that $\lim_{c\to\infty} h(c) = g/8$.

SOLUTION Write

$$h(c) = c\sqrt{c^2/g^2 + 1/4} - c^2/g = \left(c\sqrt{c^2/g^2 + 1/4} - c^2/g\right)\frac{c\sqrt{c^2/g^2 + 1/4} + c^2/g}{c\sqrt{c^2/g^2 + 1/4} + c^2/g}$$

$$= \frac{c^2(c^2/g^2 + 1/4) - c^4/g^2}{c\sqrt{c^2/g^2 + 1/4} + c^2/g} = \frac{c^2/4}{c\sqrt{c^2/g^2 + 1/4} + c^2/g}.$$

Thus,

$$\lim_{c\to\infty} h(c) = \lim_{c\to\infty} \frac{c^2/4}{c\sqrt{c^2/g^2 + 1/4} + c^2/g} = \frac{c^2/4}{2c^2/g} = \frac{g}{8}.$$

Further Insights and Challenges

45. Every limit as $x \to \infty$ can be rewritten as a one-sided limit as $t \to 0+$, where $t = x^{-1}$. Setting $g(t) = f(t^{-1})$, we have

$$\lim_{x\to\infty} f(x) = \lim_{t\to 0+} g(t)$$

Show that $\lim_{x\to\infty} \frac{3x^2 - x}{2x^2 + 5} = \lim_{t\to 0+} \frac{3 - t}{2 + 5t^2}$, and evaluate using the Quotient Law.

SOLUTION Let $t = x^{-1}$. Then $x = t^{-1}$, $t \to 0+$ as $x \to \infty$, and

$$\frac{3x^2 - x}{2x^2 + 5} = \frac{3t^{-2} - t^{-1}}{2t^{-2} + 5} = \frac{3 - t}{2 + 5t^2}.$$

Thus,

$$\lim_{x\to\infty} \frac{3x^2 - x}{2x^2 + 5} = \lim_{t\to 0+} \frac{3 - t}{2 + 5t^2} = \frac{3}{2}.$$

46. Rewrite the following as one-sided limits as in Exercise 45 and evaluate.

(a) $\lim_{x\to\infty} \frac{3 - 12x^3}{4x^3 + 3x + 1}$

(b) $\lim_{x\to\infty} e^{1/x}$

(c) $\lim_{x\to\infty} x \sin \frac{1}{x}$

(d) $\lim_{x\to\infty} \ln\left(\frac{x + 1}{x - 1}\right)$

SOLUTION

(a) Let $t = x^{-1}$. Then $x = t^{-1}$, $t \to 0+$ as $x \to \infty$, and

$$\frac{3 - 12x^3}{4x^3 + 3x + 1} = \frac{3 - 12t^{-3}}{4t^{-3} + 3t^{-1} + 1} = \frac{3t^3 - 12}{4 + 3t^2 + t^3}.$$

Thus,

$$\lim_{x\to\infty} \frac{3 - 12x^3}{4x^3 + 3x + 1} = \lim_{t\to 0+} \frac{3t^3 - 12}{4 + 3t^2 + t^3} = \frac{-12}{4} = -3.$$

(b) Let $t = x^{-1}$. Then $x = t^{-1}$, $t \to 0+$ as $x \to \infty$, and $e^{1/x} = e^t$. Thus,

$$\lim_{x\to\infty} e^{1/x} = \lim_{t\to 0+} e^t = e^0 = 1.$$

(c) Let $t = x^{-1}$. Then $x = t^{-1}$, $t \to 0+$ as $x \to \infty$, and

$$x \sin \frac{1}{x} = \frac{1}{t} \sin t = \frac{\sin t}{t}.$$

Thus,

$$\lim_{x\to\infty} x \sin \frac{1}{x} = \lim_{t\to 0+} \frac{\sin t}{t} = 1.$$

(d) Let $t = x^{-1}$. Then $x = t^{-1}$, $t \to 0+$ as $x \to \infty$, and

$$\frac{x + 1}{x - 1} = \frac{t^{-1} + 1}{t^{-1} - 1} = \frac{1 + t}{1 - t}.$$

Thus,

$$\lim_{x\to\infty} \ln\left(\frac{x + 1}{x - 1}\right) = \lim_{t\to 0+} \ln\left(\frac{1 + t}{1 - t}\right) = \ln 1 = 0.$$

47. Let $G(b) = \lim_{x\to\infty} (1 + b^x)^{1/x}$ for $b \geq 0$. Investigate $G(b)$ numerically and graphically for $b = 0.2, 0.8, 2, 3, 5$ (and additional values if necessary). Then make a conjecture for the value of $G(b)$ as a function of b. Draw a graph of $y = G(b)$. Does $G(b)$ appear to be continuous? We will evaluate $G(b)$ using L'Hôpital's Rule in Section 4.5 (see Exercise 69 in Section 4.5).

SOLUTION

- $b = 0.2$:

x	5	10	50	100
$f(x)$	1.000064	1.000000	1.000000	1.000000

It appears that $G(0.2) = 1$.

- $b = 0.8$:

x	5	10	50	100
$f(x)$	1.058324	1.010251	1.000000	1.000000

It appears that $G(0.8) = 1$.

- $b = 2$:

x	5	10	50	100
$f(x)$	2.012347	2.000195	2.000000	2.000000

It appears that $G(2) = 2$.

- $b = 3$:

x	5	10	50	100
$f(x)$	3.002465	3.000005	3.000000	3.000000

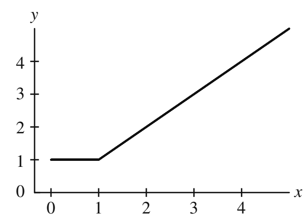
It appears that $G(3) = 3$.

- $b = 5$:

x	5	10	50	100
$f(x)$	5.000320	5.000000	5.000000	5.000000

It appears that $G(5) = 5$.

Based on these observations we conjecture that $G(b) = 1$ if $0 \le b \le 1$ and $G(b) = b$ for $b > 1$. The graph of $y = G(b)$ is shown below; the graph does appear to be continuous.



2.8 Intermediate Value Theorem

Preliminary Questions

1. Prove that $f(x) = x^2$ takes on the value 0.5 in the interval $[0, 1]$.

SOLUTION Observe that $f(x) = x^2$ is continuous on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. Because $f(0) < 0.5 < f(1)$, the Intermediate Value Theorem guarantees there is a $c \in [0, 1]$ such that $f(c) = 0.5$.

2. The temperature in Vancouver was 8°C at 6 AM and rose to 20°C at noon. Which assumption about temperature allows us to conclude that the temperature was 15°C at some moment of time between 6 AM and noon?

SOLUTION We must assume that temperature is a continuous function of time.

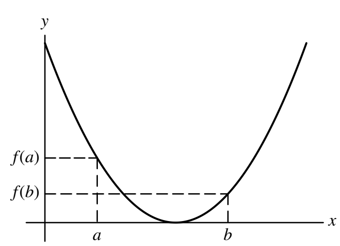
3. What is the graphical interpretation of the IVT?

SOLUTION If f is continuous on $[a, b]$, then the horizontal line $y = k$ for every k between $f(a)$ and $f(b)$ intersects the graph of $y = f(x)$ at least once.

4. Show that the following statement is false by drawing a graph that provides a counterexample:

If $f(x)$ is continuous and has a root in $[a, b]$, then $f(a)$ and $f(b)$ have opposite signs.

SOLUTION



5. Assume that $f(t)$ is continuous on $[1, 5]$ and that $f(1) = 20$, $f(5) = 100$. Determine whether each of the following statements is always true, never true, or sometimes true.

(a) $f(c) = 3$ has a solution with $c \in [1, 5]$.

(b) $f(c) = 75$ has a solution with $c \in [1, 5]$.

(c) $f(c) = 50$ has no solution with $c \in [1, 5]$.

(d) $f(c) = 30$ has exactly one solution with $c \in [1, 5]$.

SOLUTION

(a) This statement is sometimes true.

(b) This statement is always true.

(c) This statement is never true.

(d) This statement is sometimes true.

Exercises

1. Use the IVT to show that $f(x) = x^3 + x$ takes on the value 9 for some x in $[1, 2]$.

SOLUTION Observe that $f(1) = 2$ and $f(2) = 10$. Since f is a polynomial, it is continuous everywhere; in particular on $[1, 2]$. Therefore, by the IVT there is a $c \in [1, 2]$ such that $f(c) = 9$.

2. Show that $g(t) = \dfrac{t}{t+1}$ takes on the value 0.499 for some t in $[0, 1]$.

SOLUTION $g(0) = 0$ and $g(1) = \frac{1}{2}$. Since $g(t)$ is continuous for all $x \neq -1$, and since $0 < 0.4999 < \frac{1}{2}$, the IVT states that $g(t) = 0.4999$ for some t between 0 and 1.

3. Show that $g(t) = t^2 \tan t$ takes on the value $\frac{1}{2}$ for some t in $\left[0, \frac{\pi}{4}\right]$.

SOLUTION $g(0) = 0$ and $g(\frac{\pi}{4}) = \frac{\pi^2}{16}$. $g(t)$ is continuous for all t between 0 and $\frac{\pi}{4}$, and $0 < \frac{1}{2} < \frac{\pi^2}{16}$; therefore, by the IVT, there is a $c \in [0, \frac{\pi}{4}]$ such that $g(c) = \frac{1}{2}$.

4. Show that $f(x) = \dfrac{x^2}{x^7 + 1}$ takes on the value 0.4.

SOLUTION $f(0) = 0 < 0.4$. $f(1) = \frac{1}{2} > 0.4$. $f(x)$ is continuous at all points x where $x \neq -1$, therefore $f(x) = 0.4$ for some x between 0 and 1.

5. Show that $\cos x = x$ has a solution in the interval $[0, 1]$. *Hint:* Show that $f(x) = x - \cos x$ has a zero in $[0, 1]$.

SOLUTION Let $f(x) = x - \cos x$. Observe that f is continuous with $f(0) = -1$ and $f(1) = 1 - \cos 1 \approx 0.46$. Therefore, by the IVT there is a $c \in [0, 1]$ such that $f(c) = c - \cos c = 0$. Thus $c = \cos c$ and hence the equation $\cos x = x$ has a solution c in $[0, 1]$.

6. Use the IVT to find an interval of length $\frac{1}{2}$ containing a root of $f(x) = x^3 + 2x + 1$.

SOLUTION Let $f(x) = x^3 + 2x + 1$. Observe that $f(-1) = -2$ and $f(0) = 1$. Since f is continuous, we may conclude by the IVT that f has a root in $[-1, 0]$. Now, $f(-\frac{1}{2}) = -\frac{1}{8}$ so $f(-\frac{1}{2})$ and $f(0)$ are of opposite sign. Therefore, the IVT guarantees that f has a root on $[-\frac{1}{2}, 0]$.

In Exercises 7–16, prove using the IVT.

7. $\sqrt{c} + \sqrt{c+2} = 3$ has a solution.

SOLUTION Let $f(x) = \sqrt{x} + \sqrt{x+2} - 3$. Note that f is continuous on $\left[\frac{1}{4}, 2\right]$ with $f(\frac{1}{4}) = \sqrt{\frac{1}{4}} + \sqrt{\frac{9}{4}} - 3 = -1$ and $f(2) = \sqrt{2} - 1 \approx 0.41$. Therefore, by the IVT there is a $c \in \left[\frac{1}{4}, 2\right]$ such that $f(c) = \sqrt{c} + \sqrt{c+2} - 3 = 0$. Thus $\sqrt{c} + \sqrt{c+2} = 3$ and hence the equation $\sqrt{x} + \sqrt{x+2} = 3$ has a solution c in $\left[\frac{1}{4}, 2\right]$.

8. For all integers n, $\sin nx = \cos x$ for some $x \in [0, \pi]$.

SOLUTION For each integer n, let $f(x) = \sin nx - \cos x$. Observe that f is continuous with $f(0) = -1$ and $f(\pi) = 1$. Therefore, by the IVT there is a $c \in [0, \pi]$ such that $f(c) = \sin nc - \cos c = 0$. Thus $\sin nc = \cos c$ and hence the equation $\sin nx = \cos x$ has a solution c in the interval $[0, \pi]$.

9. $\sqrt{2}$ exists. *Hint:* Consider $f(x) = x^2$.

SOLUTION Let $f(x) = x^2$. Observe that f is continuous with $f(1) = 1$ and $f(2) = 4$. Therefore, by the IVT there is a $c \in [1, 2]$ such that $f(c) = c^2 = 2$. This proves the existence of $\sqrt{2}$, a number whose square is 2.

10. A positive number c has an nth root for all positive integers n.

SOLUTION If $c = 1$, then $\sqrt[n]{c} = 1$. Now, suppose $c \neq 1$. Let $f(x) = x^n - c$, and let $b = \max\{1, c\}$. Then, if $c > 1$, $b^n = c^n > c$, and if $c < 1$, $b^n = 1 > c$. So $b^n > c$. Now observe that $f(0) = -c < 0$ and $f(b) = b^n - c > 0$. Since f is continuous on $[0, b]$, by the intermediate value theorem, there is some $d \in [0, b]$ such that $f(d) = 0$. We can refer to d as $\sqrt[n]{c}$.

11. For all positive integers k, $\cos x = x^k$ has a solution.

SOLUTION For each positive integer k, let $f(x) = x^k - \cos x$. Observe that f is continuous on $\left[0, \frac{\pi}{2}\right]$ with $f(0) = -1$ and $f(\frac{\pi}{2}) = \left(\frac{\pi}{2}\right)^k > 0$. Therefore, by the IVT there is a $c \in \left[0, \frac{\pi}{2}\right]$ such that $f(c) = c^k - \cos(c) = 0$. Thus $\cos c = c^k$ and hence the equation $\cos x = x^k$ has a solution c in the interval $\left[0, \frac{\pi}{2}\right]$.

12. $2^x = bx$ has a solution if $b > 2$.

SOLUTION Let $f(x) = 2^x - bx$. Observe that f is continuous on $[0, 1]$ with $f(0) = 1 > 0$ and $f(1) = 2 - b < 0$. Therefore, by the IVT, there is a $c \in [0, 1]$ such that $f(c) = 2^c - bc = 0$.

13. $2^x + 3^x = 4^x$ has a solution.

SOLUTION Let $f(x) = 2^x + 3^x - 4^x$. Observe that f is continuous on $[0, 2]$ with $f(0) = 1 > 0$ and $f(2) = -3 < 0$. Therefore, by the IVT, there is a $c \in (0, 2)$ such that $f(c) = 2^c + 3^c - 4^c = 0$.

14. $\cos x = \cos^{-1} x$ has a solution in $(0, 1)$.

SOLUTION Let $f(x) = \cos x - \cos^{-1} x$. Observe that f is continuous on $[0, 1]$ with $f(0) = 1 - \frac{\pi}{2} < 0$ and $f(1) = \cos 1 - 0 \approx 0.54 > 0$. Therefore, by the IVT, there is a $c \in (0, 1)$ such that $f(c) = \cos c - \cos^{-1} c = 0$.

15. $e^x + \ln x = 0$ has a solution.

SOLUTION Let $f(x) = e^x + \ln x$. Observe that f is continuous on $[e^{-2}, 1]$ with $f(e^{-2}) = e^{e^{-2}} - 2 < 0$ and $f(1) = e > 0$. Therefore, by the IVT, there is a $c \in (e^{-2}, 1) \subset (0, 1)$ such that $f(c) = e^c + \ln c = 0$.

16. $\tan^{-1} x = \cos^{-1} x$ has a solution.

SOLUTION Let $f(x) = \tan^{-1} x - \cos^{-1} x$. Observe that f is continuous on $[0, 1]$ with $f(0) = \tan^{-1} 0 - \cos^{-1} 0 = -\frac{\pi}{2} < 0$ and $f(1) = \tan^{-1} 1 - \cos^{-1} 1 = \frac{\pi}{4} > 0$. Therefore, by the IVT, there is a $c \in (0, 1)$ such that $f(c) = \tan^{-1} c - \cos^{-1} c = 0$.

17. Carry out three steps of the Bisection Method for $f(x) = 2^x - x^3$ as follows:

(a) Show that $f(x)$ has a zero in $[1, 1.5]$.

(b) Show that $f(x)$ has a zero in $[1.25, 1.5]$.

(c) Determine whether $[1.25, 1.375]$ or $[1.375, 1.5]$ contains a zero.

SOLUTION Note that $f(x)$ is continuous for all x.

(a) $f(1) = 1$, $f(1.5) = 2^{1.5} - (1.5)^3 < 3 - 3.375 < 0$. Hence, $f(x) = 0$ for some x between 1 and 1.5.

(b) $f(1.25) \approx 0.4253 > 0$ and $f(1.5) < 0$. Hence, $f(x) = 0$ for some x between 1.25 and 1.5.

(c) $f(1.375) \approx -0.0059$. Hence, $f(x) = 0$ for some x between 1.25 and 1.375.

18. Figure 1 shows that $f(x) = x^3 - 8x - 1$ has a root in the interval $[2.75, 3]$. Apply the Bisection Method twice to find an interval of length $\frac{1}{16}$ containing this root.

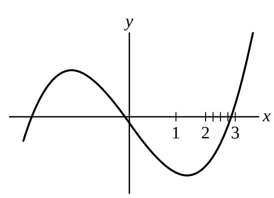


FIGURE 1 Graph of $y = x^3 - 8x - 1$.

SOLUTION Let $f(x) = x^3 - 8x - 1$. Observe that f is continuous with $f(2.75) = -2.203125$ and $f(3) = 2$. Therefore, by the IVT there is a $c \in [2.75, 3]$ such that $f(c) = 0$. The midpoint of the interval $[2.75, 3]$ is 2.875 and $f(2.875) = -0.236$. Hence, $f(x) = 0$ for some x between 2.875 and 3. The midpoint of the interval $[2.875, 3]$ is 2.9375 and $f(2.9375) = 0.84$. Thus, $f(x) = 0$ for some x between 2.875 and 2.9375.

19. Find an interval of length $\frac{1}{4}$ in $[1, 2]$ containing a root of the equation $x^7 + 3x - 10 = 0$.

SOLUTION Let $f(x) = x^7 + 3x - 10$. Observe that f is continuous with $f(1) = -6$ and $f(2) = 124$. Therefore, by the IVT there is a $c \in [1, 2]$ such that $f(c) = 0$. $f(1.5) \approx 11.59 > 0$, so $f(c) = 0$ for some $c \in [1, 1.5]$. $f(1.25) \approx -1.48 < 0$, and so $f(c) = 0$ for some $c \in [1.25, 1.5]$. This means that $[1.25, 1.5]$ is an interval of length 0.25 containing a root of $f(x)$.

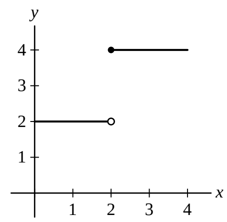
20. Show that $\tan^3 \theta - 8 \tan^2 \theta + 17 \tan \theta - 8 = 0$ has a root in $[0.5, 0.6]$. Apply the Bisection Method twice to find an interval of length 0.025 containing this root.

SOLUTION Let $f(x) = \tan^3\theta - 8\tan^2\theta + 17\tan\theta - 8$. Since $f(0.5) = -0.937387 < 0$ and $f(0.6) = 0.206186 > 0$, we conclude that $f(x) = 0$ has a root in $[0.5, 0.6]$. Since $f(0.55) = -0.35393 < 0$ and $f(0.6) > 0$, we can conclude that $f(x) = 0$ has a root in $[0.55, 0.6]$. Since $f(0.575) = -0.0707752 < 0$, we can conclude that f has a root on $[0.575, 0.6]$.

In Exercises 21–24, draw the graph of a function $f(x)$ on $[0, 4]$ with the given property.

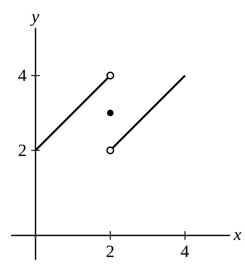
21. Jump discontinuity at $x = 2$ and does not satisfy the conclusion of the IVT.

SOLUTION The function graphed below has a jump discontinuity at $x = 2$. Note that while $f(0) = 2$ and $f(4) = 4$, there is no point c in the interval $[0, 4]$ such that $f(c) = 3$. Accordingly, the conclusion of the IVT is *not* satisfied.



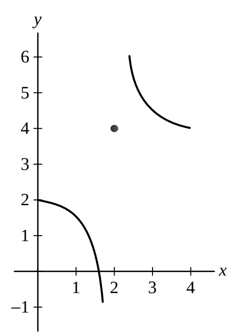
22. Jump discontinuity at $x = 2$ and satisfies the conclusion of the IVT on $[0, 4]$.

SOLUTION The function graphed below has a jump discontinuity at $x = 2$. Note that for every value M between $f(0) = 2$ and $f(4) = 4$, there *is* a point c in the interval $[0, 4]$ such that $f(c) = M$. Accordingly, the conclusion of the IVT *is* satisfied.



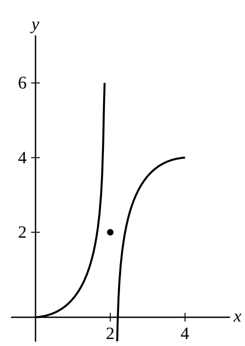
23. Infinite one-sided limits at $x = 2$ and does not satisfy the conclusion of the IVT.

SOLUTION The function graphed below has infinite one-sided limits at $x = 2$. Note that while $f(0) = 2$ and $f(4) = 4$, there is no point c in the interval $[0, 4]$ such that $f(c) = 3$. Accordingly, the conclusion of the IVT is *not* satisfied.



24. Infinite one-sided limits at $x = 2$ and satisfies the conclusion of the IVT on $[0, 4]$.

SOLUTION The function graphed below has infinite one-sided limits at $x = 2$. Note that for every value M between $f(0) = 0$ and $f(4) = 4$, there *is* a point c in the interval $[0, 4]$ such that $f(c) = M$. Accordingly, the conclusion of the IVT *is* satisfied.



25. Can Corollary 2 be applied to $f(x) = x^{-1}$ on $[-1, 1]$? Does $f(x)$ have any roots?

SOLUTION No, because $f(x) = x^{-1}$ is not continuous on $[-1, 1]$. Even though $f(-1) = -1 < 0$ and $f(1) = 1 > 0$, the function has no roots between $x = -1$ and $x = 1$. In fact, this function has no roots at all.

Further Insights and Challenges

26. Take any map and draw a circle on it anywhere (Figure 2). Prove that at any moment in time there exists a pair of diametrically opposite points A and B on that circle corresponding to locations where the temperatures at that moment are equal. *Hint:* Let θ be an angular coordinate along the circle and let $f(\theta)$ be the difference in temperatures at the locations corresponding to θ and $\theta + \pi$.

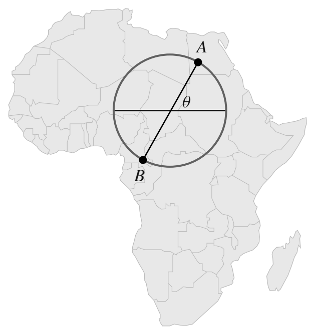


FIGURE 2 $f(\theta)$ is the difference between the temperatures at A and B.

SOLUTION Say the circle has (fixed but arbitrary) radius r and use polar coordinates with the pole at the center of the circle. For $0 \le \theta \le 2\pi$, let $T(\theta)$ be the temperature at the point $(r\cos\theta, r\sin\theta)$. We assume this temperature varies continuously. For $0 \le \theta \le \pi$, define f as the difference $f(\theta) = T(\theta) - T(\theta + \pi)$. Then f is continuous on $[0, \pi]$. There are three cases.

- If $f(0) = T(0) - T(\pi) = 0$, then $T(0) = T(\pi)$ and we have found a pair of diametrically opposite points on the circle at which the temperatures are equal.
- If $f(0) = T(0) - T(\pi) > 0$, then

$$f(\pi) = T(\pi) - T(2\pi) = T(\pi) - T(0) < 0.$$

  [Note that the angles 0 and 2π correspond to the same point, $(x, y) = (r, 0)$.] Since f is continuous on $[0, \pi]$, we have by the IVT that $f(c) = T(c) - T(c + \pi) = 0$ for some $c \in [0, \pi]$. Accordingly, $T(c) = T(c + \pi)$ and we have again found a pair of diametrically opposite points on the circle at which the temperatures are equal.
- If $f(0) = T(0) - T(\pi) < 0$, then

$$f(\pi) = T(\pi) - T(2\pi) = T(\pi) - T(0) > 0.$$

  Since f is continuous on $[0, \pi]$, we have by the IVT that $f(d) = T(d) - T(d + \pi) = 0$ for some $d \in [0, \pi]$. Accordingly, $T(d) = T(d + \pi)$ and once more we have found a pair of diametrically opposite points on the circle at which the temperatures are equal.

CONCLUSION: There is always a pair of diametrically opposite points on the circle at which the temperatures are equal.

27. Show that if $f(x)$ is continuous and $0 \le f(x) \le 1$ for $0 \le x \le 1$, then $f(c) = c$ for some c in $[0, 1]$ (Figure 3).

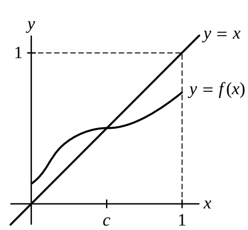


FIGURE 3 A function satisfying $0 \le f(x) \le 1$ for $0 \le x \le 1$.

SOLUTION If $f(0) = 0$, the proof is done with $c = 0$. We may assume that $f(0) > 0$. Let $g(x) = f(x) - x$. $g(0) = f(0) - 0 = f(0) > 0$. Since $f(x)$ is continuous, the Rule of Differences dictates that $g(x)$ is continuous. We need to prove that $g(c) = 0$ for some $c \in [0, 1]$. Since $f(1) \le 1$, $g(1) = f(1) - 1 \le 0$. If $g(1) = 0$, the proof is done with $c = 1$, so let's assume that $g(1) < 0$.

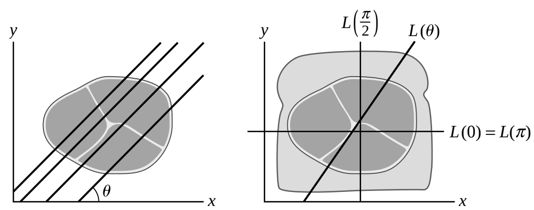
We now have a continuous function $g(x)$ on the interval $[0, 1]$ such that $g(0) > 0$ and $g(1) < 0$. From the IVT, there must be some $c \in [0, 1]$ so that $g(c) = 0$, so $f(c) - c = 0$ and so $f(c) = c$.

This is a simple case of a very general, useful, and beautiful theorem called the **Brouwer fixed point theorem**.

28. Use the IVT to show that if $f(x)$ is continuous and one-to-one on an interval $[a, b]$, then $f(x)$ is either an increasing or a decreasing function.

SOLUTION Let $f(x)$ be a continuous, one-to-one function on the interval $[a, b]$. Suppose for sake of contradiction that $f(x)$ is neither increasing nor decreasing on $[a, b]$. Now, $f(x)$ cannot be constant for that would contradict the condition that $f(x)$ is one-to-one. It follows that somewhere on $[a, b]$, $f(x)$ must transition from increasing to decreasing or from decreasing to increasing. To be specific, suppose $f(x)$ is increasing for $x_1 < x < x_2$ and decreasing for $x_2 < x < x_3$. Let k be any number between $\max\{f(x_1), f(x_3)\}$ and $f(x_2)$. Because $f(x)$ is continuous, the IVT guarantees there exists a $c_1 \in (x_1, x_2)$ such that $f(c_1) = k$; moreover, there exists a $c_2 \in (x_2, x_3)$ such that $f(c_2) = k$. However, this contradicts the condition that $f(x)$ is one-to-one. A similar analysis for the case when $f(x)$ is decreasing for $x_1 < x < x_2$ and increasing for $x_2 < x < x_3$ again leads to a contradiction. Therefore, $f(x)$ must either be increasing or decreasing on $[a, b]$.

29. **Ham Sandwich Theorem** Figure 4(A) shows a slice of ham. Prove that for any angle θ $(0 \le \theta \le \pi)$, it is possible to cut the slice in half with a cut of incline θ. *Hint:* The lines of inclination θ are given by the equations $y = (\tan\theta)x + b$, where b varies from $-\infty$ to ∞. Each such line divides the slice into two pieces (one of which may be empty). Let $A(b)$ be the amount of ham to the left of the line minus the amount to the right, and let A be the total area of the ham. Show that $A(b) = -A$ if b is sufficiently large and $A(b) = A$ if b is sufficiently negative. Then use the IVT. This works if $\theta \ne 0$ or $\frac{\pi}{2}$. If $\theta = 0$, define $A(b)$ as the amount of ham above the line $y = b$ minus the amount below. How can you modify the argument to work when $\theta = \frac{\pi}{2}$ (in which case $\tan\theta = \infty$)?



(A) Cutting a slice of ham at an angle θ.

(B) A slice of ham on top of a slice of bread.

FIGURE 4

SOLUTION Let θ be such that $\theta \ne \frac{\pi}{2}$. For any b, consider the line $L(\theta)$ drawn at angle θ to the x axis starting at $(0, b)$. This line has formula $y = (\tan\theta)x + b$. Let $A(b)$ be the amount of ham above the line minus that below the line.

Let $A > 0$ be the area of the ham. We have to accept the following (reasonable) assumptions:

- For low enough $b = b_0$, the line $L(\theta)$ lies entirely below the ham, so that $A(b_0) = A - 0 = A$.
- For high enough b_1, the line $L(\theta)$ lies entirely above the ham, so that $A(b_1) = 0 - A = -A$.
- $A(b)$ is continuous as a function of b.

Under these assumptions, we see $A(b)$ is a continuous function satisfying $A(b_0) > 0$ and $A(b_1) < 0$ for some $b_0 < b_1$. By the IVT, $A(b) = 0$ for some $b \in [b_0, b_1]$.

Suppose that $\theta = \frac{\pi}{2}$. Let the line $L(c)$ be the vertical line through $(c, 0)$ $(x = c)$. Let $A(c)$ be the area of ham to the left of $L(c)$ minus that to the right of $L(c)$. Since $L(0)$ lies entirely to the left of the ham, $A(0) = 0 - A = -A$. For some $c = c_1$ sufficiently large, $L(c)$ lies entirely to the right of the ham, so that $A(c_1) = A - 0 = A$. Hence $A(c)$ is a continuous function of c such that $A(0) < 0$ and $A(c_1) > 0$. By the IVT, there is some $c \in [0, c_1]$ such that $A(c) = 0$.

30. Figure 4(B) shows a slice of ham on a piece of bread. Prove that it is possible to slice this open-faced sandwich so that each part has equal amounts of ham and bread. *Hint:* By Exercise 29, for all $0 \le \theta \le \pi$ there is a line $L(\theta)$ of incline θ (which we assume is unique) that divides the ham into two equal pieces. Let $B(\theta)$ denote the amount of bread to the left of (or above) $L(\theta)$ minus the amount to the right (or below). Notice that $L(\pi)$ and $L(0)$ are the same line, but $B(\pi) = -B(0)$ since left and right get interchanged as the angle moves from 0 to π. Assume that $B(\theta)$ is continuous and apply the IVT. (By a further extension of this argument, one can prove the full "Ham Sandwich Theorem," which states that if you allow the knife to cut at a slant, then it is possible to cut a sandwich consisting of a slice of ham and two slices of bread so that all three layers are divided in half.)

SOLUTION For each angle θ, $0 \le \theta < \pi$, let $L(\theta)$ be the line at angle θ to the x-axis that slices the ham exactly in half, as shown in Figure 4. Let $L(0) = L(\pi)$ be the horizontal line cutting the ham in half, also as shown. For θ and $L(\theta)$ thus defined, let $B(\theta) =$ the amount of bread to the left of $L(\theta)$ minus that to the right of $L(\theta)$.

To understand this argument, one must understand what we mean by "to the left" or "to the right". Here, we mean to the left or right of the line as viewed in the direction θ. Imagine you are walking along the line in direction θ (directly right if $\theta = 0$, directly left if $\theta = \pi$, etc).

We will further accept the fact that B is continuous as a function of θ, which seems intuitively obvious. We need to prove that $B(c) = 0$ for some angle c.

Since $L(0)$ and $L(\pi)$ are drawn in opposite direction, $B(0) = -B(\pi)$. If $B(0) > 0$, we apply the IVT on $[0, \pi]$ with $B(0) > 0$, $B(\pi) < 0$, and B continuous on $[0, \pi]$; by IVT, $B(c) = 0$ for some $c \in [0, \pi]$. On the other hand, if $B(0) < 0$, then we apply the IVT with $B(0) < 0$ and $B(\pi) > 0$. If $B(0) = 0$, we are also done; $L(0)$ is the appropriate line.

2.9 The Formal Definition of a Limit

Preliminary Questions

1. Given that $\lim_{x\to 0} \cos x = 1$, which of the following statements is true?

(a) If $|\cos x - 1|$ is very small, then x is close to 0.

(b) There is an $\epsilon > 0$ such that $|x| < 10^{-5}$ if $0 < |\cos x - 1| < \epsilon$.

(c) There is a $\delta > 0$ such that $|\cos x - 1| < 10^{-5}$ if $0 < |x| < \delta$.

(d) There is a $\delta > 0$ such that $|\cos x| < 10^{-5}$ if $0 < |x - 1| < \delta$.

SOLUTION The true statement is **(c)**: There is a $\delta > 0$ such that $|\cos x - 1| < 10^{-5}$ if $0 < |x| < \delta$.

2. Suppose it is known that for a given ϵ and δ, $|f(x) - 2| < \epsilon$ if $0 < |x - 3| < \delta$. Which of the following statements must also be true?

(a) $|f(x) - 2| < \epsilon$ if $0 < |x - 3| < 2\delta$

(b) $|f(x) - 2| < 2\epsilon$ if $0 < |x - 3| < \delta$

(c) $|f(x) - 2| < \dfrac{\epsilon}{2}$ if $0 < |x - 3| < \dfrac{\delta}{2}$

(d) $|f(x) - 2| < \epsilon$ if $0 < |x - 3| < \dfrac{\delta}{2}$

SOLUTION Statements **(b)** and **(d)** are true.

Exercises

1. Based on the information conveyed in Figure 1(A), find values of L, ϵ, and $\delta > 0$ such that the following statement holds: $|f(x) - L| < \epsilon$ if $|x| < \delta$.

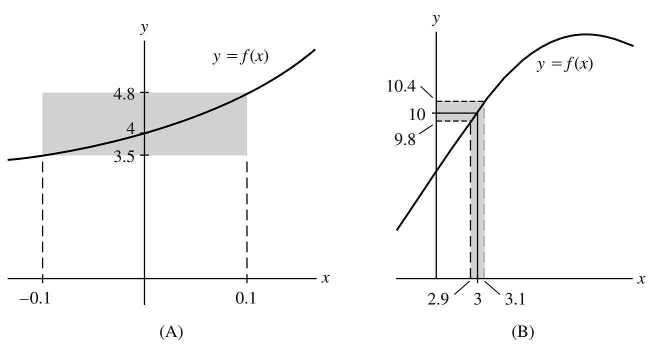


FIGURE 1

SOLUTION We see $-0.1 < x < 0.1$ forces $3.5 < f(x) < 4.8$. Rewritten, this means that $|x - 0| < 0.1$ implies that $|f(x) - 4| < 0.8$. Replacing numbers where appropriate in the definition of the limit $|x - c| < \delta$ implies $|f(x) - L| < \epsilon$, we get $L = 4$, $\epsilon = 0.8$, $c = 0$, and $\delta = 0.1$.

2. Based on the information conveyed in Figure 1(B), find values of c, L, ϵ, and $\delta > 0$ such that the following statement holds: $|f(x) - L| < \epsilon$ if $0 < |x - c| < \delta$.

SOLUTION From the shaded region in the graph, we can see that $9.8 < f(x) < 10.4$ whenever $2.9 < x < 3.1$. Rewriting these double inequalities as absolute value inequalities, we get $|f(x) - 10| < 0.4$ whenever $0 < |x - 3| < 0.1$. Replacing numbers where appropriate in the definition of the limit $0 < |x - c| < \delta$ implies $|f(x) - L| < \epsilon$, we get $L = 10$, $\epsilon = 0.4$, $c = 3$, and $\delta = 0.1$.

3. Consider $\lim_{x\to 4} f(x)$, where $f(x) = 8x + 3$.

(a) Show that $|f(x) - 35| = 8|x - 4|$.

(b) Show that for any $\epsilon > 0$, $|f(x) - 35| < \epsilon$ if $0 < |x - 4| < \delta$, where $\delta = \frac{\epsilon}{8}$. Explain how this proves rigorously that $\lim_{x\to 4} f(x) = 35$.

SOLUTION

(a) $|f(x) - 35| = |8x + 3 - 35| = |8x - 32| = |8(x - 4)| = 8\,|x - 4|$. (Remember that the last step is justified because $8 > 0$).

(b) Let $\epsilon > 0$. Let $\delta = \epsilon/8$ and suppose $0 < |x - 4| < \delta$. By part **(a)**, $|f(x) - 35| = 8|x - 4| < 8\delta$. Substituting $\delta = \epsilon/8$, we see $|f(x) - 35| < 8\epsilon/8 = \epsilon$. We see that, for any $\epsilon > 0$, we found an appropriate δ so that $0 < |x - 4| < \delta$ implies $|f(x) - 35| < \epsilon$. Hence $\lim_{x\to 4} f(x) = 35$.

4. Consider $\lim_{x\to 2} f(x)$, where $f(x) = 4x - 1$.

(a) Show that $|f(x) - 7| < 4\delta$ if $0 < |x - 2| < \delta$.

(b) Find a δ such that

$$|f(x) - 7| < 0.01 \qquad \text{if} \qquad 0 < |x - 2| < \delta$$

(c) Prove rigorously that $\lim_{x\to 2} f(x) = 7$.

SOLUTION

(a) If $0 < |x - 2| < \delta$, then $|(4x - 1) - 7| = 4|x - 2| < 4\delta$.

(b) If $0 < |x - 2| < \delta = 0.0025$, then $|(4x - 1) - 7| = 4|x - 2| < 4\delta = 0.01$.

(c) Let $\epsilon > 0$ be given. Then whenever $0 < |x - 2| < \delta = \epsilon/4$, we have $|(4x - 1) - 7| = 4|x - 2| < 4\delta = \epsilon$. Since ϵ was arbitrary, we conclude that $\lim_{x\to 2}(4x - 1) = 7$.

5. Consider $\lim_{x\to 2} x^2 = 4$ (refer to Example 2).

(a) Show that $|x^2 - 4| < 0.05$ if $0 < |x - 2| < 0.01$.

(b) Show that $|x^2 - 4| < 0.0009$ if $0 < |x - 2| < 0.0002$.

(c) Find a value of δ such that $|x^2 - 4|$ is less than 10^{-4} if $0 < |x - 2| < \delta$.

SOLUTION

(a) If $0 < |x - 2| < \delta = 0.01$, then $|x| < 3$ and $\left|x^2 - 4\right| = |x - 2||x + 2| \le |x - 2|\,(|x| + 2) < 5|x - 2| < 0.05$.

(b) If $0 < |x - 2| < \delta = 0.0002$, then $|x| < 2.0002$ and

$$\left|x^2 - 4\right| = |x - 2||x + 2| \le |x - 2|\,(|x| + 2) < 4.0002|x - 2| < 0.00080004 < 0.0009.$$

(c) Note that $\left|x^2 - 4\right| = |(x + 2)(x - 2)| \le |x + 2|\,|x - 2|$. Since $|x - 2|$ can get arbitrarily small, we can require $|x - 2| < 1$ so that $1 < x < 3$. This ensures that $|x + 2|$ is at most 5. Now we know that $\left|x^2 - 4\right| \le 5|x - 2|$. Let $\delta = 10^{-5}$. Then, if $0 < |x - 2| < \delta$, we get $\left|x^2 - 4\right| \le 5|x - 2| < 5 \times 10^{-5} < 10^{-4}$ as desired.

6. With regard to the limit $\lim_{x\to 5} x^2 = 25$,

(a) Show that $|x^2 - 25| < 11|x - 5|$ if $4 < x < 6$. *Hint:* Write $|x^2 - 25| = |x + 5| \cdot |x - 5|$.

(b) Find a δ such that $|x^2 - 25| < 10^{-3}$ if $0 < |x - 5| < \delta$.

(c) Give a rigorous proof of the limit by showing that $|x^2 - 25| < \epsilon$ if $0 < |x - 5| < \delta$, where δ is the smaller of $\frac{\epsilon}{11}$ and 1.

SOLUTION

(a) If $4 < x < 6$, then $|x - 5| < \delta = 1$ and $\left|x^2 - 25\right| = |x - 5||x + 5| \le |x - 5|\,(|x| + 5) < 11|x - 5|$.

(b) If $0 < |x - 5| < \delta = \frac{0.001}{11}$, then $x < 6$ and $\left|x^2 - 25\right| = |x - 5||x + 5| \le |x - 5|\,(|x| + 5) < 11|x - 5| < 0.001$.

(c) Let $0 < |x - 5| < \delta = \min\left\{1, \frac{\epsilon}{11}\right\}$. Since $\delta < 1$, $|x - 5| < \delta < 1$ implies $4 < x < 6$. Specifically, $x < 6$ and

$$\left|x^2 - 25\right| = |x - 5||x + 5| \le |x - 5|\,(|x| + 5) < |x - 5|(6 + 5) = 11|x - 5|.$$

Since δ is also less than $\epsilon/11$, we can conclude $11|x - 5| < 11(\epsilon/11) = \epsilon$, thus completing the rigorous proof that $|x^2 - 25| < \epsilon$ if $|x - 5| < \delta$.

7. Refer to Example 3 to find a value of $\delta > 0$ such that

$$\left|\frac{1}{x} - \frac{1}{3}\right| < 10^{-4} \qquad \text{if} \qquad 0 < |x - 3| < \delta$$

SOLUTION The Example shows that for any $\epsilon > 0$ we have

$$\left|\frac{1}{x} - \frac{1}{3}\right| \le \epsilon \quad \text{if } 0 < |x - 3| < \delta$$

where δ is the smaller of the numbers 6ϵ and 1. In our case, we may take $\delta = 6 \times 10^{-4}$.

8. Use Figure 2 to find a value of $\delta > 0$ such that the following statement holds: $\left|1/x^2 - \frac{1}{4}\right| < \epsilon$ if $0 < |x - 2| < \delta$ for $\epsilon = 0.03$. Then find a value of δ that works for $\epsilon = 0.01$.

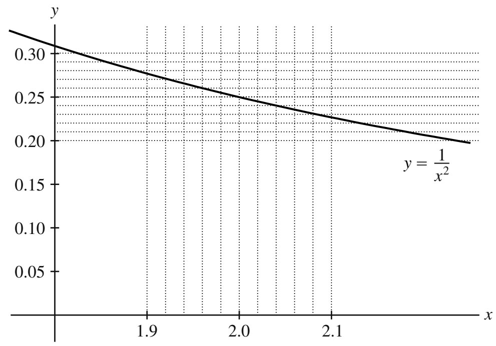


FIGURE 2

SOLUTION From Figure 2, we see that $0.22 < \frac{1}{x^2} < 0.28$ for $1.9 < x < 2.1$. Rewriting these expressions using absolute values yields

$$\left|\frac{1}{x^2} - \frac{1}{4}\right| < 0.03$$

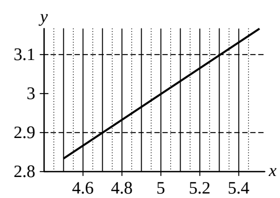
for $0 < |x - 2| < 0.1$. Thus, for $\epsilon = 0.03$, we may take $\delta = 0.1$. Additionally, we see that $0.24 < \frac{1}{x^2} < 0.26$ for $1.96 < x < 2.04$. Rewriting these expressions using absolute values yields

$$\left|\frac{1}{x^2} - \frac{1}{4}\right| < 0.01$$

for $0 < |x - 2| < 0.04$. Thus, for $\epsilon = 0.01$, we may take $\delta = 0.04$.

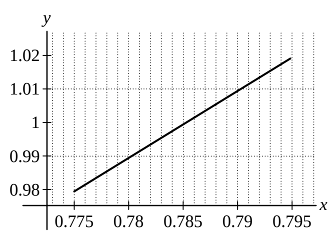
9. GU Plot $f(x) = \sqrt{2x - 1}$ together with the horizontal lines $y = 2.9$ and $y = 3.1$. Use this plot to find a value of $\delta > 0$ such that $|\sqrt{2x - 1} - 3| < 0.1$ if $|x - 5| < \delta$.

SOLUTION From the plot below, we see that $\delta = 0.25$ will guarantee that $|\sqrt{2x - 1} - 3| < 0.1$ whenever $|x - 5| \le \delta$.



10. GU Plot $f(x) = \tan x$ together with the horizontal lines $y = 0.99$ and $y = 1.01$. Use this plot to find a value of $\delta > 0$ such that $|\tan x - 1| < 0.01$ if $\left|x - \frac{\pi}{4}\right| < \delta$.

SOLUTION From the plot below, we see that $\delta = 0.005$ will guarantee that $|\tan x - 1| < 0.01$ whenever $|x - \frac{\pi}{4}| \le \delta$.

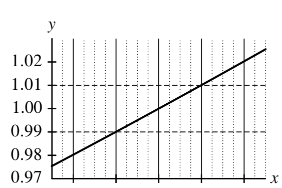


11. GU The number e has the following property: $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$. Use a plot of $f(x) = \frac{e^x - 1}{x}$ to find a value of $\delta > 0$ such that $|f(x) - 1| < 0.01$ if $|x - 1| < \delta$.

SOLUTION From the plot below, we see that $\delta = 0.02$ will guarantee that

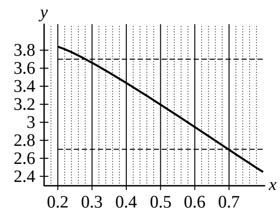
$$\left|\frac{e^x - 1}{x} - 1\right| < 0.01$$

whenever $|x| < \delta$.

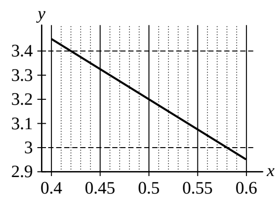


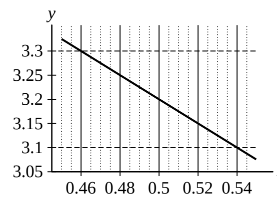
12. GU Let $f(x) = \dfrac{4}{x^2+1}$ and $\epsilon = 0.5$. Using a plot of $f(x)$, find a value of $\delta > 0$ such that $\left|f(x) - \frac{16}{5}\right| < \epsilon$ for $0 < \left|x - \frac{1}{2}\right| < \delta$. Repeat for $\epsilon = 0.2$ and 0.1.

SOLUTION From the plot below, we see that $\delta = 0.18$ will guarantee that $|f(x) - \frac{16}{5}| < 0.5$ whenever $0 < |x - \frac{1}{2}| < \delta$.



When $\epsilon = 0.2$, we see that $\delta = 0.075$ will guarantee $|f(x) - \frac{16}{5}| < \epsilon$ whenever $0 < |x - \frac{1}{2}| < \delta$ (examine the plot below at the left); when $\epsilon = 0.1$, $\delta = 0.035$ will guarantee $|f(x) - \frac{16}{5}| < \epsilon$ whenever $0 < |x - \frac{1}{2}| < \delta$ (examine the plot below at the right).





13. Consider $\lim\limits_{x\to 2} \dfrac{1}{x}$.

(a) Show that if $|x-2| < 1$, then

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{1}{2}|x-2|$$

(b) Let δ be the smaller of 1 and 2ϵ. Prove:

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \epsilon \qquad \text{if} \qquad 0 < |x-2| < \delta$$

(c) Find a $\delta > 0$ such that $\left|\frac{1}{x} - \frac{1}{2}\right| < 0.01$ if $0 < |x-2| < \delta$.

(d) Prove rigorously that $\lim\limits_{x\to 2} \dfrac{1}{x} = \dfrac{1}{2}$.

SOLUTION

(a) Since $|x-2| < 1$, it follows that $1 < x < 3$, in particular that $x > 1$. Because $x > 1$, then $\dfrac{1}{x} < 1$ and

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| = \frac{|x-2|}{2x} < \frac{1}{2}|x-2|.$$

(b) Let $\delta = \min\{1, 2\epsilon\}$ and suppose that $0 < |x-2| < \delta$. Then by part (a) we have

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{1}{2}|x-2| < \frac{1}{2}\delta < \frac{1}{2}\cdot 2\epsilon = \epsilon.$$

(c) Choose $\delta = 0.02$. Then $\left|\dfrac{1}{x} - \dfrac{1}{2}\right| < \dfrac{1}{2}\delta = 0.01$ by part (b).

(d) Let $\epsilon > 0$ be given. Then whenever $0 < |x-2| < \delta = \min\{1, 2\epsilon\}$, we have

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \frac{1}{2}\delta \le \epsilon.$$

Since ϵ was arbitrary, we conclude that $\lim\limits_{x\to 2} \dfrac{1}{x} = \dfrac{1}{2}$.

14. Consider $\lim\limits_{x\to 1} \sqrt{x+3}$.

(a) Show that $|\sqrt{x+3} - 2| < \frac{1}{2}|x-1|$ if $|x-1| < 4$. *Hint:* Multiply the inequality by $|\sqrt{x+3} + 2|$ and observe that $|\sqrt{x+3} + 2| > 2$.

(b) Find $\delta > 0$ such that $|\sqrt{x+3} - 2| < 10^{-4}$ for $0 < |x-1| < \delta$.

(c) Prove rigorously that the limit is equal to 2.

SOLUTION

(a) $|x-1|<4$ implies that $-3<x<5$. Since $x>-3$, then $\sqrt{x+3}$ is defined (and positive), whence

$$\left|\sqrt{x+3}-2\right| = \left|\frac{\left(\sqrt{x+3}-2\right)}{1}\frac{\left(\sqrt{x+3}+2\right)}{\left(\sqrt{x+3}+2\right)}\right| = \frac{|x-1|}{\sqrt{x+3}+2} < \frac{|x-1|}{2}.$$

(b) Choose $\delta = 0.0002$. Then provided $0<|x-1|<\delta$, we have $x>-3$ and therefore

$$\left|\sqrt{x+3}-2\right| < \frac{|x-1|}{2} < \frac{\delta}{2} = 0.0001$$

by part (a).

(c) Let $\epsilon > 0$ be given. Then whenever $0<|x-1|<\delta = \min\{2\epsilon, 4\}$, we have $x>-3$ and thus

$$\left|\sqrt{x+3}-2\right| = \left|\frac{\left(\sqrt{x+3}-2\right)}{1}\frac{\left(\sqrt{x+3}+2\right)}{\left(\sqrt{x+3}+2\right)}\right| = \frac{|x-1|}{\sqrt{x+3}+2} < \frac{2\epsilon}{2} = \epsilon.$$

Since ϵ was arbitrary, we conclude that $\lim_{x\to 1}\sqrt{x+3} = 2$.

15. Let $f(x) = \sin x$. Using a calculator, we find:

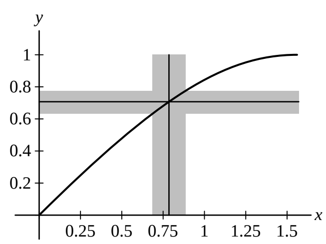
$$f\left(\frac{\pi}{4}-0.1\right) \approx 0.633, \quad f\left(\frac{\pi}{4}\right) \approx 0.707, \quad f\left(\frac{\pi}{4}+0.1\right) \approx 0.774$$

Use these values and the fact that $f(x)$ is increasing on $\left[0, \frac{\pi}{2}\right]$ to justify the statement

$$\left|f(x) - f\left(\frac{\pi}{4}\right)\right| < 0.08 \qquad \text{if} \qquad 0 < \left|x - \frac{\pi}{4}\right| < 0.1$$

Then draw a figure like Figure 3 to illustrate this statement.

SOLUTION Since $f(x)$ is increasing on the interval, the three $f(x)$ values tell us that $0.633 \le f(x) \le 0.774$ for all x between $\frac{\pi}{4}-0.1$ and $\frac{\pi}{4}+0.1$. We may subtract $f(\frac{\pi}{4})$ from the inequality for $f(x)$. This show that, for $\frac{\pi}{4}-0.1 < x < \frac{\pi}{4}+0.1$, $0.633 - f(\frac{\pi}{4}) \le f(x) - f(\frac{\pi}{4}) \le 0.774 - f(\frac{\pi}{4})$. This means that, if $0<|x-\frac{\pi}{4}|<0.1$, then $0.633 - 0.707 \le f(x) - f(\frac{\pi}{4}) \le 0.774 - 0.707$, so $-0.074 \le f(x) - f(\frac{\pi}{4}) \le 0.067$. Then $-0.08 < f(x) - f(\frac{\pi}{4}) < 0.08$ follows from this, so $0<|x-\frac{\pi}{4}|<0.1$ implies $|f(x) - f(\frac{\pi}{4})| < 0.08$. The figure below illustrates this.



16. Adapt the argument in Example 1 to prove rigorously that $\lim_{x\to c}(ax+b) = ac+b$, where a, b, c are arbitrary.

SOLUTION $|f(x) - (ac+b)| = |(ax+b)-(ac+b)| = |a(x-c)| = |a|\,|x-c|$. This says the gap is $|a|$ times as large as $|x-c|$. Let $\epsilon > 0$. Let $\delta = \epsilon/|a|$. If $|x-c|<\delta$, we get $|f(x)-(ac+b)| = |a|\,|x-c| < |a|\epsilon/|a| = \epsilon$, which is what we had to prove.

17. Adapt the argument in Example 2 to prove rigorously that $\lim_{x\to c} x^2 = c^2$ for all c.

SOLUTION To relate the gap to $|x-c|$, we take

$$\left|x^2 - c^2\right| = |(x+c)(x-c)| = |x+c|\,|x-c|\,.$$

We choose δ in two steps. First, since we are requiring $|x-c|$ to be small, we require $\delta < |c|$, so that x lies between 0 and $2c$. This means that $|x+c| < 3|c|$, so $|x-c||x+c| < 3|c|\delta$. Next, we require that $\delta < \dfrac{\epsilon}{3|c|}$, so

$$|x-c||x+c| < \frac{\epsilon}{3|c|}3|c| = \epsilon,$$

and we are done.

Therefore, given $\epsilon > 0$, we let

$$\delta = \min\left\{|c|, \frac{\epsilon}{3|c|}\right\}.$$

Then, for $|x-c|<\delta$, we have

$$|x^2-c^2| = |x-c|\,|x+c| < 3|c|\delta < 3|c|\frac{\epsilon}{3|c|} = \epsilon.$$

18. Adapt the argument in Example 3 to prove rigorously that $\lim_{x\to c} x^{-1} = \frac{1}{c}$ for all $c \neq 0$.

SOLUTION Suppose that $c \neq 0$. To relate the gap to $|x-c|$, we find:

$$\left|x^{-1}-\frac{1}{c}\right| = \left|\frac{c-x}{cx}\right| = \frac{|x-c|}{|cx|}$$

Since $|x-c|$ is required to be small, we may assume from the outset that $|x-c|<|c|/2$, so that x is between $|c|/2$ and $3|c|/2$. This forces $|cx|>|c|/2$, from which

$$\frac{|x-c|}{|cx|} < \frac{2}{|c|}|x-c|.$$

If $\delta < \epsilon(\frac{|c|}{2})$,

$$\left|x^{-1}-\frac{1}{c}\right| < \frac{2}{|c|}|x-c| < \frac{2}{|c|}\frac{|c|}{2}\epsilon = \epsilon.$$

Therefore, given $\epsilon>0$ we let

$$\delta = \min\left(\frac{|c|}{2}, \epsilon\left(\frac{|c|}{2}\right)\right).$$

We have shown that $|x^{-1}-\frac{1}{c}|<\epsilon$ if $0<|x-c|<\delta$.

In Exercises 19–24, use the formal definition of the limit to prove the statement rigorously.

19. $\lim_{x\to 4}\sqrt{x} = 2$

SOLUTION Let $\epsilon>0$ be given. We bound $|\sqrt{x}-2|$ by multiplying $\dfrac{\sqrt{x}+2}{\sqrt{x}+2}$.

$$|\sqrt{x}-2| = \left|\sqrt{x}-2\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right)\right| = \left|\frac{x-4}{\sqrt{x}+2}\right| = |x-4|\left|\frac{1}{\sqrt{x}+2}\right|.$$

We can assume $\delta<1$, so that $|x-4|<1$, and hence $\sqrt{x}+2>\sqrt{3}+2>3$. This gives us

$$|\sqrt{x}-2| = |x-4|\left|\frac{1}{\sqrt{x}+2}\right| < |x-4|\frac{1}{3}.$$

Let $\delta = \min(1, 3\epsilon)$. If $|x-4|<\delta$,

$$|\sqrt{x}-2| = |x-4|\left|\frac{1}{\sqrt{x}+2}\right| < |x-4|\frac{1}{3} < \delta\frac{1}{3} < 3\epsilon\frac{1}{3} = \epsilon,$$

thus proving the limit rigorously.

20. $\lim_{x\to 1}(3x^2+x) = 4$

SOLUTION Let $\epsilon>0$ be given. We bound $\left|(3x^2+x)-4\right|$ using quadratic factoring.

$$\left|(3x^2+x)-4\right| = \left|3x^2+x-4\right| = |(3x+4)(x-1)| = |x-1||3x+4|.$$

Let $\delta = \min(1, \frac{\epsilon}{10})$. Since $\delta<1$, we get $|3x+4|<10$, so that

$$\left|(3x^2+x)-4\right| = |x-1||3x+4| < 10|x-1|.$$

Since $\delta < \frac{\epsilon}{10}$, we get

$$\left|(3x^2+x)-4\right| < 10|x-1| < 10\frac{\epsilon}{10} = \epsilon.$$

21. $\lim_{x\to 1} x^3 = 1$

SOLUTION Let $\epsilon > 0$ be given. We bound $\left|x^3 - 1\right|$ by factoring the difference of cubes:

$$\left|x^3 - 1\right| = \left|(x^2 + x + 1)(x - 1)\right| = |x - 1| \left|x^2 + x + 1\right|.$$

Let $\delta = \min(1, \frac{\epsilon}{7})$, and assume $|x - 1| < \delta$. Since $\delta < 1$, $0 < x < 2$. Since $x^2 + x + 1$ increases as x increases for $x > 0$, $x^2 + x + 1 < 7$ for $0 < x < 2$, and so

$$\left|x^3 - 1\right| = |x - 1| \left|x^2 + x + 1\right| < 7|x - 1| < 7\frac{\epsilon}{7} = \epsilon$$

and the limit is rigorously proven.

22. $\lim_{x \to 0} (x^2 + x^3) = 0$

SOLUTION Let $\epsilon > 0$ be given. Now,

$$|(x^2 + x^3) - 0| = |x|\,|x|\,|x + 1|.$$

Let $\delta = \min(1, \frac{1}{2}\epsilon)$, and suppose $|x| < \delta$. Since $\delta < 1$, $|x| < 1$, so $-1 < x < 1$. This means $|1 + x| < 2$, so that $|x|\,|x + 1| < 2$. Thus,

$$\left|(x^2 + x^3) - 0\right| = |x|\,|x|\,|x + 1| < 2|x| < 2 \cdot \frac{1}{2}\epsilon = \epsilon.$$

and the limit is rigorously proven.

23. $\lim_{x \to 2} x^{-2} = \frac{1}{4}$

SOLUTION Let $\epsilon > 0$ be given. First, we bound $x^{-2} - \frac{1}{4}$:

$$\left|x^{-2} - \frac{1}{4}\right| = \left|\frac{4 - x^2}{4x^2}\right| = |2 - x| \left|\frac{2 + x}{4x^2}\right|.$$

Let $\delta = \min(1, \frac{4}{5}\epsilon)$, and suppose $|x - 2| < \delta$. Since $\delta < 1$, $|x - 2| < 1$, so $1 < x < 3$. This means that $4x^2 > 4$ and $|2 + x| < 5$, so that $\dfrac{2 + x}{4x^2} < \frac{5}{4}$. We get:

$$\left|x^{-2} - \frac{1}{4}\right| = |2 - x| \left|\frac{2 + x}{4x^2}\right| < \frac{5}{4}|x - 2| < \frac{5}{4} \cdot \frac{4}{5}\epsilon = \epsilon.$$

and the limit is rigorously proven.

24. $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

SOLUTION Let $\epsilon > 0$ be given. Let $\delta = \epsilon$, and assume $|x - 0| = |x| < \delta$. We bound $x \sin \frac{1}{x}$.

$$\left|x \sin \frac{1}{x} - 0\right| = |x| \left|\sin \frac{1}{x}\right| < |x| < \delta = \epsilon.$$

25. Let $f(x) = \dfrac{x}{|x|}$. Prove rigorously that $\lim_{x \to 0} f(x)$ does not exist. *Hint:* Show that for any L, there always exists some x such that $|x| < \delta$ but $|f(x) - L| \geq \frac{1}{2}$, no matter how small δ is taken.

SOLUTION Let L be any real number. Let $\delta > 0$ be any small positive number. Let $x = \frac{\delta}{2}$, which satisfies $|x| < \delta$, and $f(x) = 1$. We consider two cases:

- $(|f(x) - L| \geq \frac{1}{2})$: we are done.
- $(|f(x) - L| < \frac{1}{2})$: This means $\frac{1}{2} < L < \frac{3}{2}$. In this case, let $x = -\frac{\delta}{2}$. $f(x) = -1$, and so $\frac{3}{2} < L - f(x)$.

In either case, there exists an x such that $|x| < \frac{\delta}{2}$, but $|f(x) - L| \geq \frac{1}{2}$.

26. Prove rigorously that $\lim_{x \to 0} |x| = 0$.

SOLUTION Let $\epsilon > 0$ be given and take $\delta = \epsilon$. Then, whenever $|x| < \delta$,

$$||x| - 0| = |x| < \delta = \epsilon,$$

thus proving the limit rigorously.

27. Let $f(x) = \min(x, x^2)$, where $\min(a, b)$ is the minimum of a and b. Prove rigorously that $\lim_{x\to 1} f(x) = 1$.

SOLUTION Let $\epsilon > 0$ and let $\delta = \min(1, \frac{\epsilon}{2})$. Then, whenever $|x - 1| < \delta$, it follows that $0 < x < 2$. If $1 < x < 2$, then $\min(x, x^2) = x$ and

$$|f(x) - 1| = |x - 1| < \delta < \frac{\epsilon}{2} < \epsilon.$$

On the other hand, if $0 < x < 1$, then $\min(x, x^2) = x^2$, $|x + 1| < 2$ and

$$|f(x) - 1| = |x^2 - 1| = |x - 1|\,|x + 1| < 2\delta < \epsilon.$$

Thus, whenever $|x - 1| < \delta$, $|f(x) - 1| < \epsilon$.

28. Prove rigorously that $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist.

SOLUTION Let $\delta > 0$ be a given small positive number, and let L be any real number. We will prove that $\left|\sin \frac{1}{x} - L\right| \geq \frac{1}{2}$ for some x such that $|x| < \delta$.

Let $N > 0$ be a positive integer large enough so that $\frac{2}{(4N+1)\pi} < \delta$. Let

$$x_1 = \frac{2}{(4N + 1)\pi},$$

$$x_2 = \frac{2}{(4N + 3)\pi}.$$

$$x_2 < x_1 < \delta.$$

$$\sin \frac{1}{x_1} = \sin \frac{(4N + 1)\pi}{2} = 1 \quad \text{and} \quad \sin \frac{1}{x_2} = \sin \frac{(4N + 3)\pi}{2} = -1.$$

If $|\sin \frac{1}{x_1} - L| \geq \frac{1}{2}$, we are done. Therefore, let's assume that $|\sin \frac{1}{x_1} - L| < \frac{1}{2}$. $-\frac{1}{2} < \sin \frac{1}{x_1} - L < \frac{1}{2}$, so $L - \frac{1}{2} < \sin \frac{1}{x_1} = 1 < L + \frac{1}{2}$. This means $L > \frac{1}{2}$, so that $|\sin \frac{1}{x_2} - L| = |-1 - L| > \frac{3}{2}$. In either case, there is an x such that $|x| < \delta$ but $|\sin \frac{1}{x} - L| \geq \frac{1}{2}$, so no limit L can exist.

29. First, use the identity

$$\sin x + \sin y = 2 \sin \left(\frac{x + y}{2}\right) \cos \left(\frac{x - y}{2}\right)$$

to verify the relation

$$\sin(a + h) - \sin a = h \frac{\sin(h/2)}{h/2} \cos \left(a + \frac{h}{2}\right) \qquad \boxed{6}$$

Then use the inequality $\left|\frac{\sin x}{x}\right| \leq 1$ for $x \neq 0$ to show that $|\sin(a + h) - \sin a| < |h|$ for all a. Finally, prove rigorously that $\lim_{x\to a} \sin x = \sin a$.

SOLUTION We first write

$$\sin(a + h) - \sin a = \sin(a + h) + \sin(-a).$$

Applying the identity with $x = a + h$, $y = -a$, yields:

$$\sin(a + h) - \sin a = \sin(a + h) + \sin(-a) = 2 \sin \left(\frac{a + h - a}{2}\right) \cos \left(\frac{2a + h}{2}\right)$$

$$= 2 \sin \left(\frac{h}{2}\right) \cos \left(a + \frac{h}{2}\right) = 2 \left(\frac{h}{h}\right) \sin \left(\frac{h}{2}\right) \cos \left(a + \frac{h}{2}\right) = h \frac{\sin(h/2)}{h/2} \cos \left(a + \frac{h}{2}\right).$$

Therefore,

$$|\sin(a + h) - \sin a| = |h| \left|\frac{\sin(h/2)}{h/2}\right| \left|\cos \left(a + \frac{h}{2}\right)\right|.$$

Using the fact that $\left|\frac{\sin \theta}{\theta}\right| < 1$ and that $|\cos \theta| \leq 1$, and making the substitution $h = x - a$, we see that this last relation is equivalent to

$$|\sin x - \sin a| < |x - a|.$$

Now, to prove the desired limit, let $\epsilon > 0$, and take $\delta = \epsilon$. If $|x - a| < \delta$, then

$$|\sin x - \sin a| < |x - a| < \delta = \epsilon,$$

Therefore, a δ was found for arbitrary ϵ, and the proof is complete.

Further Insights and Challenges

30. Uniqueness of the Limit Prove that a function converges to at most one limiting value. In other words, use the limit definition to prove that if $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$, then $L_1 = L_2$.

SOLUTION Let $\epsilon > 0$ be given. Since $\lim_{x\to c} f(x) = L_1$, there exists δ_1 such that if $|x - c| < \delta_1$ then $|f(x) - L_1| < \epsilon$. Similarly, since $\lim_{x\to c} f(x) = L_2$, there exists δ_2 such that if $|x - c| < \delta_2$ then $|f(x) - L_2| < \epsilon$. Now let $|x - c| < \min(\delta_1, \delta_2)$ and observe that

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\ &\le |L_1 - f(x)| + |f(x) - L_2| \\ &= |f(x) - L_1| + |f(x) - L_2| < 2\epsilon. \end{aligned}$$

So, $|L_1 - L_2| < 2\epsilon$ for any $\epsilon > 0$. We have $|L_1 - L_2| = \lim_{\epsilon\to 0} |L_1 - L_2| < \lim_{\epsilon\to 0} 2\epsilon = 0$. Therefore, $|L_1 - L_2| = 0$ and, hence, $L_1 = L_2$.

In Exercises 31–33, prove the statement using the formal limit definition.

31. The Constant Multiple Law [Theorem 1, part (ii) in Section 2.3, p. 77]

SOLUTION Suppose that $\lim_{x\to c} f(x) = L$. We wish to prove that $\lim_{x\to c} af(x) = aL$.

Let $\epsilon > 0$ be given. $\epsilon/|a|$ is also a positive number. Since $\lim_{x\to c} f(x) = L$, we know there is a $\delta > 0$ such that $|x - c| < \delta$ forces $|f(x) - L| < \epsilon/|a|$. Suppose $|x - c| < \delta$. $|af(x) - aL| = |a||f(x) - aL| < |a|(\epsilon/|a|) = \epsilon$, so the rule is proven.

32. The Squeeze Theorem. (Theorem 1 in Section 2.6, p. 96)

SOLUTION *Proof of the Squeeze Theorem.* Suppose that (i) the inequalities $h(x) \le f(x) \le g(x)$ hold for all x near (but not equal to) a and (ii) $\lim_{x\to a} h(x) = \lim_{x\to a} g(x) = L$. Let $\epsilon > 0$ be given.

- By (i), there exists a $\delta_1 > 0$ such that $h(x) \le f(x) \le g(x)$ whenever $0 < |x - a| < \delta_1$.
- By (ii), there exist $\delta_2 > 0$ and $\delta_3 > 0$ such that $|h(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta_2$ and $|g(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta_3$.
- Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then whenever $0 < |x - a| < \delta$ we have $L - \epsilon < h(x) \le f(x) \le g(x) < L + \epsilon$; i.e., $|f(x) - L| < \epsilon$. Since ϵ was arbitrary, we conclude that $\lim_{x\to a} f(x) = L$.

33. The Product Law [Theorem 1, part (iii) in Section 2.3, p. 77]. *Hint:* Use the identity

$$f(x)g(x) - LM = (f(x) - L)\,g(x) + L(g(x) - M)$$

SOLUTION Before we can prove the Product Law, we need to establish one preliminary result. We are given that $\lim_{x\to c} g(x) = M$. Consequently, if we set $\epsilon = 1$, then the definition of a limit guarantees the existence of a $\delta_1 > 0$ such that whenever $0 < |x - c| < \delta_1$, $|g(x) - M| < 1$. Applying the inequality $|g(x)| - |M| \le |g(x) - M|$, it follows that $|g(x)| < 1 + |M|$. In other words, because $\lim_{x\to c} g(x) = M$, there exists a $\delta_1 > 0$ such that $|g(x)| < 1 + |M|$ whenever $0 < |x - c| < \delta_1$.

We can now prove the Product Law. Let $\epsilon > 0$. As proven above, because $\lim_{x\to c} g(x) = M$, there exists a $\delta_1 > 0$ such that $|g(x)| < 1 + |M|$ whenever $0 < |x - c| < \delta_1$. Furthermore, by the definition of a limit, $\lim_{x\to c} g(x) = M$ implies there exists a $\delta_2 > 0$ such that $|g(x) - M| < \frac{\epsilon}{2(1+|L|)}$ whenever $0 < |x - c| < \delta_2$. We have included the "1+" in the denominator to avoid division by zero in case $L = 0$. The reason for including the factor of 2 in the denominator will become clear shortly. Finally, because $\lim_{x\to c} f(x) = L$, there exists a $\delta_3 > 0$ such that $|f(x) - L| < \frac{\epsilon}{2(1+|M|)}$ whenever $0 < |x - c| < \delta_3$. Now, let $\delta = \min(\delta_1, \delta_2, \delta_3)$. Then, for all x satisfying $0 < |x - c| < \delta$, we have

$$\begin{aligned} |f(x)g(x) - LM| &= |(f(x) - L)g(x) + L(g(x) - M)| \\ &\le |f(x) - L|\,|g(x)| + |L|\,|g(x) - M| \\ &< \frac{\epsilon}{2(1 + |M|)}(1 + |M|) + |L|\frac{\epsilon}{2(1 + |L|)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence,

$$\lim_{x\to c} f(x)g(x) = LM = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x).$$

34. Let $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational. Prove that $\lim_{x\to c} f(x)$ does not exist for any c.

SOLUTION Let c be any number, and let $\delta > 0$ be an arbitrary small number. We will prove that there is an x such that $|x - c| < \delta$, but $|f(x) - f(c)| > \frac{1}{2}$. c must be either irrational or rational. If c is rational, then $f(c) = 1$. Since the irrational numbers are dense, there is at least one irrational number z such that $|z - c| < \delta$. $|f(z) - f(c)| = 1 > \frac{1}{2}$, so the function is discontinuous at $x = c$. On the other hand, if c is irrational, then there is a *rational* number q such that $|q - c| < \delta$. $|f(q) - f(c)| = |1 - 0| = 1 > \frac{1}{2}$, so the function is discontinuous at $x = c$.

35. Here is a function with strange continuity properties:

$$f(x) = \begin{cases} \dfrac{1}{q} & \text{if } x \text{ is the rational number } p/q \text{ in lowest terms} \\ 0 & \text{if } x \text{ is an irrational number} \end{cases}$$

(a) Show that $f(x)$ is discontinuous at c if c is rational. *Hint:* There exist irrational numbers arbitrarily close to c.
(b) Show that $f(x)$ is continuous at c if c is irrational. *Hint:* Let I be the interval $\{x : |x - c| < 1\}$. Show that for any $Q > 0$, I contains at most finitely many fractions p/q with $q < Q$. Conclude that there is a δ such that all fractions in $\{x : |x - c| < \delta\}$ have a denominator larger than Q.

SOLUTION

(a) Let c be any rational number and suppose that, in lowest terms, $c = p/q$, where p and q are integers. To prove the discontinuity of f at c, we must show there is an $\epsilon > 0$ such that for any $\delta > 0$ there is an x for which $|x - c| < \delta$, but that $|f(x) - f(c)| > \epsilon$. Let $\epsilon = \frac{1}{2q}$ and $\delta > 0$. Since there is at least one irrational number between any two distinct real numbers, there is some irrational x between c and $c + \delta$. Hence, $|x - c| < \delta$, but $|f(x) - f(c)| = |0 - \frac{1}{q}| = \frac{1}{q} > \frac{1}{2q} = \epsilon$.

(b) Let c be irrational, let $\epsilon > 0$ be given, and let $N > 0$ be a prime integer sufficiently large so that $\frac{1}{N} < \epsilon$. Let $\frac{p_1}{q_1}, \dots, \frac{p_m}{q_m}$ be all rational numbers $\frac{p}{q}$ in lowest terms such that $|\frac{p}{q} - c| < 1$ and $q < N$. Since N is finite, this is a finite list; hence, one number $\frac{p_i}{q_i}$ in the list must be closest to c. Let $\delta = \frac{1}{2}|\frac{p_i}{q_i} - c|$. By construction, $|\frac{p_i}{q_i} - c| > \delta$ for all $i = 1 \dots m$. Therefore, for any rational number $\frac{p}{q}$ such that $|\frac{p}{q} - c| < \delta$, $q > N$, so $\frac{1}{q} < \frac{1}{N} < \epsilon$.

Therefore, for any *rational* number x such that $|x - c| < \delta$, $|f(x) - f(c)| < \epsilon$. $|f(x) - f(c)| = 0$ for any irrational number x, so $|x - c| < \delta$ implies that $|f(x) - f(c)| < \epsilon$ for any number x.

CHAPTER REVIEW EXERCISES

1. The position of a particle at time t (s) is $s(t) = \sqrt{t^2 + 1}$ m. Compute its average velocity over $[2, 5]$ and estimate its instantaneous velocity at $t = 2$.

SOLUTION Let $s(t) = \sqrt{t^2 + 1}$. The average velocity over $[2, 5]$ is

$$\frac{s(5) - s(2)}{5 - 2} = \frac{\sqrt{26} - \sqrt{5}}{3} \approx 0.954 \text{ m/s}.$$

From the data in the table below, we estimate that the instantaneous velocity at $t = 2$ is approximately 0.894 m/s.

interval	[1.9, 2]	[1.99, 2]	[1.999, 2]	[2, 2.001]	[2, 2.01]	[2, 2.1]
average ROC	0.889769	0.893978	0.894382	0.894472	0.894873	0.898727

2. The "wellhead" price p of natural gas in the United States (in dollars per 1000 ft^3) on the first day of each month in 2008 is listed in the table below.

J	F	M	A	M	J
6.99	7.55	8.29	8.94	9.81	10.82

J	A	S	O	N	D
10.62	8.32	7.27	6.36	5.97	5.87

Compute the average rate of change of p (in dollars per 1000 ft^3 per month) over the quarterly periods January–March, April–June, and July–September.

SOLUTION To determine the average rate of change in price over the first quarter, divide the difference between the April and January prices by the three-month duration of the quarter. This yields

$$\frac{8.94 - 6.99}{3} = 0.65 \text{ dollars per 1000 ft}^3 \text{ per month.}$$

In a similar manner, we calculate the average rates of change for the second and third quarters of the year to be

$$\frac{10.62 - 8.94}{3} = 0.56 \text{ dollars per 1000 ft}^3 \text{ per month.}$$

and

$$\frac{6.36 - 10.62}{3} = -1.42 \text{ dollars per 1000 ft}^3 \text{ per month.}$$

3. For a whole number n, let $P(n)$ be the number of *partitions* of n, that is, the number of ways of writing n as a sum of one or more whole numbers. For example, $P(4) = 5$ since the number 4 can be partitioned in five different ways: 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$, and $1 + 1 + 1 + 1$. Treating $P(n)$ as a continuous function, use Figure 1 to estimate the rate of change of $P(n)$ at $n = 12$.

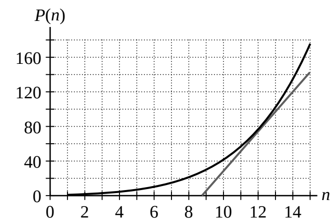


FIGURE 1 Graph of $P(n)$.

SOLUTION The tangent line drawn in the figure appears to pass through the points (15, 140) and (10.5, 40). We therefore estimate that the rate of change of $P(n)$ at $n = 12$ is

$$\frac{140 - 40}{15 - 10.5} = \frac{100}{4.5} = \frac{200}{9}.$$

4. The average velocity v (m/s) of an oxygen molecule in the air at temperature T (°C) is $v = 25.7\sqrt{273.15 + T}$. What is the average speed at $T = 25°$ (room temperature)? Estimate the rate of change of average velocity with respect to temperature at $T = 25°$. What are the units of this rate?

SOLUTION Let $v(T) = 25.7\sqrt{273.15 + T}$. The average velocity at $T = 25°\text{C}$ is

$$v(25) = 25.7\sqrt{273.15 + 25} \approx 443.76 \text{ m/s}.$$

From the data in the table below, we estimate that the rate of change of velocity with respect to temperature when $T = 25°\text{C}$ is 0.7442 m/s^2.

interval	[24.9, 25]	[24.99, 25]	[24.999, 25]	[25, 25.001]	[25, 25.01]	[25, 25.1]
average ROC	0.744256	0.744199	0.744193	0.744195	0.744187	0.744131

In Exercises 5–10, estimate the limit numerically to two decimal places or state that the limit does not exist.

5. $\lim_{x\to 0} \frac{1 - \cos^3(x)}{x^2}$

SOLUTION Let $f(x) = \frac{1-\cos^3 x}{x^2}$. The data in the table below suggests that

$$\lim_{x\to 0} \frac{1 - \cos^3 x}{x^2} \approx 1.50.$$

In constructing the table, we take advantage of the fact that f is an even function.

x	± 0.001	± 0.01	± 0.1
$f(x)$	1.500000	1.499912	1.491275

(The exact value is $\frac{3}{2}$.)

6. $\lim_{x\to 1} x^{1/(x-1)}$

SOLUTION Let $f(x) = x^{1/(x-1)}$. The data in the table below suggests that

$$\lim_{x\to 1} x^{1/(x-1)} \approx 2.72.$$

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	2.867972	2.731999	2.719642	2.716924	2.704814	2.593742

(The exact value is e.)

7. $\lim_{x\to 2} \frac{x^x - 4}{x^2 - 4}$

SOLUTION Let $f(x) = \frac{x^x-4}{x^2-4}$. The data in the table below suggests that

$$\lim_{x\to 2} \frac{x^x - 4}{x^2 - 4} \approx 1.69.$$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	1.575461	1.680633	1.691888	1.694408	1.705836	1.828386

(The exact value is $1 + \ln 2$.)

8. $\lim_{x\to 2} \frac{x - 2}{\ln(3x - 5)}$

SOLUTION Let $f(x) = \frac{x-2}{\ln(3x-5)}$. The data in the table below suggests that

$$\lim_{x\to 2} \frac{x - 2}{\ln(3x - 5)} \approx 0.33.$$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.280367	0.328308	0.332833	0.333833	0.338309	0.381149

(The exact value is $1/3$.)

9. $\lim_{x\to 1} \left(\frac{7}{1 - x^7} - \frac{3}{1 - x^3}\right)$

SOLUTION Let $f(x) = \left(\frac{7}{1-x^7} - \frac{3}{1-x^3}\right)$. The data in the table below suggests that

$$\lim_{x\to 1} \left(\frac{7}{1 - x^7} - \frac{3}{1 - x^3}\right) \approx 2.00.$$

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	2.347483	2.033498	2.003335	1.996668	1.966835	1.685059

(The exact value is 2.)

10. $\lim_{x\to 2} \frac{3^x - 9}{5^x - 25}$

SOLUTION Let $f(x) = \frac{3^x-9}{5^x-25}$. The data in the table below suggests that

$$\lim_{x\to 2} \frac{3^x - 9}{5^x - 25} \approx 0.246.$$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.251950	0.246365	0.245801	0.245675	0.245110	0.239403

(The exact value is $\frac{9}{25} \frac{\ln 3}{\ln 5}$.)

In Exercises 11–50, evaluate the limit if it exists. If not, determine whether the one-sided limits exist (finite or infinite).

11. $\lim_{x\to 4} (3 + x^{1/2})$

SOLUTION $\lim_{x\to 4} (3 + x^{1/2}) = 3 + \sqrt{4} = 5.$

12. $\lim_{x\to 1} \frac{5 - x^2}{4x + 7}$

SOLUTION $\lim_{x\to 1} \frac{5 - x^2}{4x + 7} = \frac{5 - 1^2}{4(1) + 7} = \frac{4}{11}.$

13. $\lim_{x\to -2} \frac{4}{x^3}$

SOLUTION $\lim_{x\to -2} \dfrac{4}{x^3} = \dfrac{4}{(-2)^3} = -\dfrac{1}{2}$.

14. $\lim_{x\to -1} \dfrac{3x^2+4x+1}{x+1}$

SOLUTION $\lim_{x\to -1} \dfrac{3x^2+4x+1}{x+1} = \lim_{x\to -1} \dfrac{(3x+1)(x+1)}{x+1} = \lim_{x\to -1}(3x+1) = 3(-1)+1 = -2.$

15. $\lim_{t\to 9} \dfrac{\sqrt{t}-3}{t-9}$

SOLUTION $\lim_{t\to 9} \dfrac{\sqrt{t}-3}{t-9} = \lim_{t\to 9} \dfrac{\sqrt{t}-3}{(\sqrt{t}-3)(\sqrt{t}+3)} = \lim_{t\to 9} \dfrac{1}{\sqrt{t}+3} = \dfrac{1}{\sqrt{9}+3} = \dfrac{1}{6}$.

16. $\lim_{x\to 3} \dfrac{\sqrt{x+1}-2}{x-3}$

SOLUTION

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x\to 3} \frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)}$$
$$= \lim_{x\to 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{\sqrt{3+1}+2} = \frac{1}{4}.$$

17. $\lim_{x\to 1} \dfrac{x^3-x}{x-1}$

SOLUTION $\lim_{x\to 1} \dfrac{x^3-x}{x-1} = \lim_{x\to 1} \dfrac{x(x-1)(x+1)}{x-1} = \lim_{x\to 1} x(x+1) = 1(1+1) = 2.$

18. $\lim_{h\to 0} \dfrac{2(a+h)^2-2a^2}{h}$

SOLUTION

$$\lim_{h\to 0} \frac{2(a+h)^2-2a^2}{h} = \lim_{h\to 0} \frac{2a^2+4ah+2h^2-2a^2}{h} = \lim_{h\to 0} \frac{h(4a+2h)}{h} = \lim_{h\to 0}(4a+2h) = 4a+2(0) = 4a.$$

19. $\lim_{t\to 9} \dfrac{t-6}{\sqrt{t}-3}$

SOLUTION Because the one-sided limits

$$\lim_{t\to 9-} \frac{t-6}{\sqrt{t}-3} = -\infty \qquad \text{and} \qquad \lim_{t\to 9+} \frac{t-6}{\sqrt{t}-3} = \infty,$$

are not equal, the two-sided limit

$$\lim_{t\to 9} \frac{t-6}{\sqrt{t}-3} \qquad \text{does not exist.}$$

20. $\lim_{s\to 0} \dfrac{1-\sqrt{s^2+1}}{s^2}$

SOLUTION

$$\lim_{s\to 0} \frac{1-\sqrt{s^2+1}}{s^2} = \lim_{s\to 0} \frac{1-\sqrt{s^2+1}}{s^2} \cdot \frac{1+\sqrt{s^2+1}}{1+\sqrt{s^2+1}} = \lim_{s\to 0} \frac{1-(s^2+1)}{s^2(1+\sqrt{s^2+1})}$$
$$= \lim_{s\to 0} \frac{-1}{1+\sqrt{s^2+1}} = \frac{-1}{1+\sqrt{0^2+1}} = -\frac{1}{2}.$$

21. $\lim_{x\to -1+} \dfrac{1}{x+1}$

SOLUTION For $x > -1$, $x+1 > 0$. Therefore,

$$\lim_{x\to -1+} \frac{1}{x+1} = \infty.$$

22. $\lim_{y\to \frac{1}{3}} \dfrac{3y^2+5y-2}{6y^2-5y+1}$

SOLUTION

$$\lim_{y\to\frac{1}{3}} \frac{3y^2+5y-2}{6y^2-5y+1} = \lim_{y\to\frac{1}{3}} \frac{(3y-1)(y+2)}{(3y-1)(2y-1)} = \lim_{y\to\frac{1}{3}} \frac{y+2}{2y-1} = -7.$$

23. $\lim_{x\to 1} \frac{x^3-2x}{x-1}$

SOLUTION Because the one-sided limits

$$\lim_{x\to 1-} \frac{x^3-2x}{x-1} = \infty \qquad \text{and} \qquad \lim_{x\to 1+} \frac{x^3-2x}{x-1} = -\infty,$$

are not equal, the two-sided limit

$$\lim_{x\to 1} \frac{x^3-2x}{x-1} \qquad \text{does not exist.}$$

24. $\lim_{a\to b} \frac{a^2-3ab+2b^2}{a-b}$

SOLUTION $\lim_{a\to b} \frac{a^2-3ab+2b^2}{a-b} = \lim_{a\to b} \frac{(a-b)(a-2b)}{a-b} = \lim_{a\to b}(a-2b) = b-2b = -b.$

25. $\lim_{x\to 0} \frac{e^{3x}-e^x}{e^x-1}$

SOLUTION

$$\lim_{x\to 0} \frac{e^{3x}-e^x}{e^x-1} = \lim_{x\to 0} \frac{e^x(e^x-1)(e^x+1)}{e^x-1} = \lim_{x\to 0} e^x(e^x+1) = 1\cdot 2 = 2.$$

26. $\lim_{\theta\to 0} \frac{\sin 5\theta}{\theta}$

SOLUTION

$$\lim_{\theta\to 0} \frac{\sin 5\theta}{\theta} = 5\lim_{\theta\to 0} \frac{\sin 5\theta}{5\theta} = 5(1) = 5.$$

27. $\lim_{x\to 1.5} \frac{[x]}{x}$

SOLUTION $\lim_{x\to 1.5} \frac{[x]}{x} = \frac{[1.5]}{1.5} = \frac{1}{1.5} = \frac{2}{3}.$

28. $\lim_{\theta\to\frac{\pi}{4}} \sec\theta$

SOLUTION

$$\lim_{\theta\to\frac{\pi}{4}} \sec\theta = \sec\frac{\pi}{4} = \sqrt{2}.$$

29. $\lim_{z\to -3} \frac{z+3}{z^2+4z+3}$

SOLUTION

$$\lim_{z\to -3} \frac{z+3}{z^2+4z+3} = \lim_{z\to -3} \frac{z+3}{(z+3)(z+1)} = \lim_{z\to -3} \frac{1}{z+1} = -\frac{1}{2}.$$

30. $\lim_{x\to 1} \frac{x^3-ax^2+ax-1}{x-1}$

SOLUTION Using

$$x^3-ax^2+ax-1 = (x-1)(x^2+x+1) - ax(x-1) = (x-1)(x^2+x-ax+1)$$

we find

$$\lim_{x\to 1} \frac{x^3-ax^2+ax-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x-ax+1)}{x-1} = \lim_{x\to 1}(x^2+x-ax+1)$$
$$= 1^2+1-a(1)+1 = 3-a.$$

31. $\lim_{x\to b} \dfrac{x^3-b^3}{x-b}$

SOLUTION $\lim_{x\to b} \dfrac{x^3-b^3}{x-b} = \lim_{x\to b} \dfrac{(x-b)(x^2+xb+b^2)}{x-b} = \lim_{x\to b}(x^2+xb+b^2) = b^2+b(b)+b^2 = 3b^2.$

32. $\lim_{x\to 0} \dfrac{\sin 4x}{\sin 3x}$

SOLUTION

$$\lim_{x\to 0} \frac{\sin 4x}{\sin 3x} = \frac{4}{3}\lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\frac{3x}{\sin 3x} = \frac{4}{3}\lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\lim_{x\to 0}\frac{3x}{\sin 3x} = \frac{4}{3}(1)(1) = \frac{4}{3}.$$

33. $\lim_{x\to 0}\left(\dfrac{1}{3x} - \dfrac{1}{x(x+3)}\right)$

SOLUTION $\lim_{x\to 0}\left(\dfrac{1}{3x} - \dfrac{1}{x(x+3)}\right) = \lim_{x\to 0}\dfrac{(x+3)-3}{3x(x+3)} = \lim_{x\to 0}\dfrac{1}{3(x+3)} = \dfrac{1}{3(0+3)} = \dfrac{1}{9}.$

34. $\lim_{\theta\to\frac{1}{4}} 3^{\tan(\pi\theta)}$

SOLUTION

$$\lim_{\theta\to\frac{1}{4}} 3^{\tan(\pi\theta)} = 3^{\tan(\pi/4)} = 3^1 = 3.$$

35. $\lim_{x\to 0-}\dfrac{[x]}{x}$

SOLUTION For x sufficiently close to zero but negative, $[x] = -1$. Therefore,

$$\lim_{x\to 0-}\frac{[x]}{x} = \lim_{x\to 0-}\frac{-1}{x} = \infty.$$

36. $\lim_{x\to 0+}\dfrac{[x]}{x}$

SOLUTION For x sufficiently close to zero but positive, $[x] = 0$. Therefore,

$$\lim_{x\to 0+}\frac{[x]}{x} = \lim_{x\to 0+}\frac{0}{x} = 0.$$

37. $\lim_{\theta\to\frac{\pi}{2}} \theta\sec\theta$

SOLUTION Because the one-sided limits

$$\lim_{\theta\to\frac{\pi}{2}-}\theta\sec\theta = \infty \qquad \text{and} \qquad \lim_{\theta\to\frac{\pi}{2}+}\theta\sec\theta = -\infty$$

are not equal, the two-sided limit

$$\lim_{\theta\to\frac{\pi}{2}}\theta\sec\theta \qquad \text{does not exist.}$$

38. $\lim_{y\to 2}\ln\left(\sin\dfrac{\pi}{y}\right)$

SOLUTION

$$\lim_{y\to 2}\ln\left(\sin\frac{\pi}{y}\right) = \ln\left(\sin\frac{\pi}{2}\right) = \ln 1 = 0.$$

39. $\lim_{\theta\to 0}\dfrac{\cos\theta - 2}{\theta}$

SOLUTION Because the one-sided limits

$$\lim_{\theta\to 0-}\frac{\cos\theta-2}{\theta} = \infty \qquad \text{and} \qquad \lim_{\theta\to 0+}\frac{\cos\theta-2}{\theta} = -\infty$$

are not equal, the two-sided limit

$$\lim_{\theta\to 0}\frac{\cos\theta-2}{\theta} \qquad \text{does not exist.}$$

40. $\lim_{x\to 4.3} \dfrac{1}{x - [x]}$

SOLUTION $\lim_{x\to 4.3} \dfrac{1}{x-[x]} = \dfrac{1}{4.3-[4.3]} = \dfrac{1}{0.3} = \dfrac{10}{3}.$

41. $\lim_{x\to 2-} \dfrac{x-3}{x-2}$

SOLUTION For x close to 2 but less than 2, $x - 3 < 0$ and $x - 2 < 0$. Therefore,

$$\lim_{x\to 2-} \frac{x-3}{x-2} = \infty.$$

42. $\lim_{t\to 0} \dfrac{\sin^2 t}{t^3}$

SOLUTION Note that

$$\frac{\sin^2 t}{t^3} = \frac{\sin t}{t} \cdot \frac{\sin t}{t} \cdot \frac{1}{t}.$$

As $t \to 0$, each factor of $\frac{\sin t}{t}$ approaches 1; however, the factor $\frac{1}{t}$ tends to $-\infty$ as $t \to 0-$ and tends to ∞ as $t \to 0+$. Consequently,

$$\lim_{t\to 0-} \frac{\sin^2 t}{t^3} = -\infty, \quad \lim_{t\to 0+} \frac{\sin^2 t}{t^3} = \infty$$

and

$$\lim_{t\to 0} \frac{\sin^2 t}{t^3} \quad \text{does not exist.}$$

43. $\lim_{x\to 1+} \left(\dfrac{1}{\sqrt{x-1}} - \dfrac{1}{\sqrt{x^2-1}} \right)$

SOLUTION $\lim_{x\to 1+} \left(\dfrac{1}{\sqrt{x-1}} - \dfrac{1}{\sqrt{x^2-1}} \right) = \lim_{x\to 1+} \dfrac{\sqrt{x+1}-1}{\sqrt{x^2-1}} = \infty.$

44. $\lim_{t\to e} \sqrt{t}(\ln t - 1)$

SOLUTION

$$\lim_{t\to e} \sqrt{t}(\ln t - 1) = \lim_{t\to e} \sqrt{t} \cdot \lim_{t\to e} (\ln t - 1) = \sqrt{e}(\ln e - 1) = 0.$$

45. $\lim_{x\to \frac{\pi}{2}} \tan x$

SOLUTION Because the one-sided limits

$$\lim_{x\to \frac{\pi}{2}-} \tan x = \infty \qquad \text{and} \qquad \lim_{x\to \frac{\pi}{2}+} \tan x = -\infty$$

are not equal, the two-sided limit

$$\lim_{x\to \frac{\pi}{2}} \tan x \qquad \text{does not exist.}$$

46. $\lim_{t\to 0} \cos \dfrac{1}{t}$

SOLUTION As $t \to 0$, $\frac{1}{t}$ grows without bound and $\cos(\frac{1}{t})$ oscillates faster and faster. Consequently,

$$\lim_{t\to 0} \cos\left(\frac{1}{t}\right) \qquad \text{does not exist.}$$

The same is true for both one-sided limits.

47. $\lim_{t\to 0+} \sqrt{t} \cos \dfrac{1}{t}$

SOLUTION For $t > 0$,

$$-1 \le \cos\left(\frac{1}{t}\right) \le 1,$$

so

$$-\sqrt{t} \le \sqrt{t}\cos\left(\frac{1}{t}\right) \le \sqrt{t}.$$

Because

$$\lim_{t\to 0+} -\sqrt{t} = \lim_{t\to 0+} \sqrt{t} = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{t\to 0+} \sqrt{t}\cos\left(\frac{1}{t}\right) = 0.$$

48. $\displaystyle\lim_{x\to 5+} \frac{x^2-24}{x^2-25}$

SOLUTION For x close to 5 but larger than 5, $x^2 - 24 > 0$ and $x^2 - 25 > 0$. Therefore,

$$\lim_{x\to 5+} \frac{x^2-24}{x^2-25} = \infty.$$

49. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{\sin x}$

SOLUTION

$$\lim_{x\to 0} \frac{\cos x - 1}{\sin x} = \lim_{x\to 0} \frac{\cos x - 1}{\sin x}\cdot\frac{\cos x + 1}{\cos x + 1} = \lim_{x\to 0} \frac{-\sin^2 x}{\sin x(\cos x + 1)} = -\lim_{x\to 0} \frac{\sin x}{\cos x + 1} = -\frac{0}{1+1} = 0.$$

50. $\displaystyle\lim_{\theta\to 0} \frac{\tan\theta - \sin\theta}{\sin^3\theta}$

SOLUTION

$$\lim_{\theta\to 0} \frac{\tan\theta - \sin\theta}{\sin^3\theta} = \lim_{\theta\to 0} \frac{\sec\theta - 1}{\sin^2\theta} = \lim_{\theta\to 0} \frac{\sec\theta - 1}{\sin^2\theta}\cdot\frac{\sec\theta + 1}{\sec\theta + 1} = \lim_{\theta\to 0} \frac{\tan^2\theta}{\sin^2\theta(\sec\theta + 1)}$$
$$= \lim_{\theta\to 0} \frac{\sec^2\theta}{\sec\theta + 1} = \frac{1}{1+1} = \frac{1}{2}.$$

51. Find the left- and right-hand limits of the function $f(x)$ in Figure 2 at $x = 0, 2, 4$. State whether $f(x)$ is left- or right-continuous (or both) at these points.

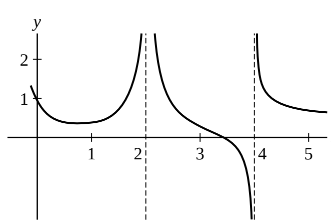


FIGURE 2

SOLUTION According to the graph of $f(x)$,

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = 1$$
$$\lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x) = \infty$$
$$\lim_{x\to 4-} f(x) = -\infty$$
$$\lim_{x\to 4+} f(x) = \infty.$$

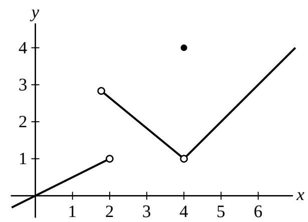
The function is both left- and right-continuous at $x = 0$ and neither left- nor right-continuous at $x = 2$ and $x = 4$.

52. Sketch the graph of a function $f(x)$ such that

(a) $\lim_{x\to 2-} f(x) = 1, \qquad \lim_{x\to 2+} f(x) = 3$

(b) $\lim_{x\to 4} f(x)$ exists but does not equal $f(4)$.

SOLUTION

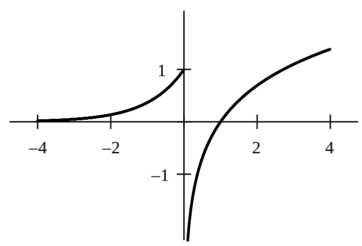


53. Graph $h(x)$ and describe the discontinuity:

$$h(x) = \begin{cases} e^x & \text{for } x \le 0 \\ \ln x & \text{for } x > 0 \end{cases}$$

Is $h(x)$ left- or right-continuous?

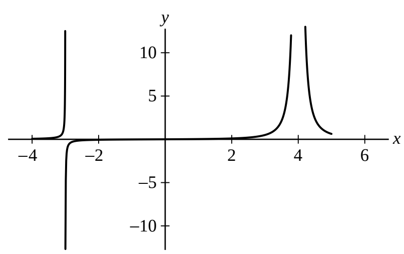
SOLUTION The graph of $h(x)$ is shown below. At $x = 0$, the function has an infinite discontinuity but is left-continuous.



54. Sketch the graph of a function $g(x)$ such that

$$\lim_{x\to -3-} g(x) = \infty, \qquad \lim_{x\to -3+} g(x) = -\infty, \qquad \lim_{x\to 4} g(x) = \infty$$

SOLUTION



55. Find the points of discontinuity of

$$g(x) = \begin{cases} \cos\left(\frac{\pi x}{2}\right) & \text{for } |x| < 1 \\ |x - 1| & \text{for } |x| \ge 1 \end{cases}$$

Determine the type of discontinuity and whether $g(x)$ is left- or right-continuous.

SOLUTION First note that $\cos\left(\frac{\pi x}{2}\right)$ is continuous for $-1 < x < 1$ and that $|x - 1|$ is continuous for $x \le -1$ and for $x \ge 1$. Thus, the only points at which $g(x)$ might be discontinuous are $x = \pm 1$. At $x = 1$, we have

$$\lim_{x\to 1-} g(x) = \lim_{x\to 1-} \cos\left(\frac{\pi x}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

and

$$\lim_{x\to 1+} g(x) = \lim_{x\to 1+} |x - 1| = |1 - 1| = 0,$$

so $g(x)$ is continuous at $x = 1$. On the other hand, at $x = -1$,

$$\lim_{x\to -1+} g(x) = \lim_{x\to -1+} \cos\left(\frac{\pi x}{2}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$$

and

$$\lim_{x\to -1-} g(x) = \lim_{x\to -1-} |x-1| = |-1-1| = 2,$$

so $g(x)$ has a jump discontinuity at $x = -1$. Since $g(-1) = 2$, $g(x)$ is left-continuous at $x = -1$.

56. Show that $f(x) = xe^{\sin x}$ is continuous on its domain.

SOLUTION Because e^x and $\sin x$ are continuous for all real numbers, their composition, $e^{\sin x}$ is continuous for all real numbers. Moreover, x is continuous for all real numbers, so the product $xe^{\sin x}$ is continuous for all real numbers. Thus, $f(x) = xe^{\sin x}$ is continuous for all real numbers.

57. Find a constant b such that $h(x)$ is continuous at $x = 2$, where

$$h(x) = \begin{cases} x+1 & \text{for } |x| < 2 \\ b - x^2 & \text{for } |x| \ge 2 \end{cases}$$

With this choice of b, find all points of discontinuity.

SOLUTION To make $h(x)$ continuous at $x = 2$, we must have the two one-sided limits as x approaches 2 be equal. With

$$\lim_{x\to 2-} h(x) = \lim_{x\to 2-} (x+1) = 2+1 = 3$$

and

$$\lim_{x\to 2+} h(x) = \lim_{x\to 2+} (b - x^2) = b - 4,$$

it follows that we must choose $b = 7$. Because $x + 1$ is continuous for $-2 < x < 2$ and $7 - x^2$ is continuous for $x \le -2$ and for $x \ge 2$, the only possible point of discontinuity is $x = -2$. At $x = -2$,

$$\lim_{x\to -2+} h(x) = \lim_{x\to -2+} (x+1) = -2+1 = -1$$

and

$$\lim_{x\to -2-} h(x) = \lim_{x\to -2-} (7 - x^2) = 7 - (-2)^2 = 3,$$

so $h(x)$ has a jump discontinuity at $x = -2$.

In Exercises 58–63, find the horizontal asymptotes of the function by computing the limits at infinity.

58. $f(x) = \dfrac{9x^2 - 4}{2x^2 - x}$

SOLUTION Because

$$\lim_{x\to\infty} \frac{9x^2-4}{2x^2-x} = \lim_{x\to\infty} \frac{9 - 4/x^2}{2 - 1/x} = \frac{9}{2}$$

and

$$\lim_{x\to -\infty} \frac{9x^2-4}{2x^2-x} = \lim_{x\to -\infty} \frac{9 - 4/x^2}{2 - 1/x} = \frac{9}{2},$$

it follows that the graph of $y = \dfrac{9x^2-4}{2x^2-x}$ has a horizontal asymptote of $\frac{9}{2}$.

59. $f(x) = \dfrac{x^2 - 3x^4}{x - 1}$

SOLUTION Because

$$\lim_{x\to\infty} \frac{x^2-3x^4}{x-1} = \lim_{x\to\infty} \frac{1/x^2 - 3}{1/x^3 - 1/x^4} = -\infty$$

and

$$\lim_{x\to -\infty} \frac{x^2-3x^4}{x-1} = \lim_{x\to -\infty} \frac{1/x^2 - 3}{1/x^3 - 1/x^4} = \infty,$$

it follows that the graph of $y = \dfrac{x^2-3x^4}{x-1}$ does not have any horizontal asymptotes.

60. $f(u) = \dfrac{8u-3}{\sqrt{16u^2+6}}$

SOLUTION Because

$$\lim_{u\to\infty} \frac{8u-3}{\sqrt{16u^2+6}} = \lim_{u\to\infty} \frac{8-3/u}{\sqrt{16+6/u^2}} = \frac{8}{\sqrt{16}} = 2$$

and

$$\lim_{u\to-\infty} \frac{8u-3}{\sqrt{16u^2+6}} = \lim_{u\to-\infty} \frac{8-3/u}{-\sqrt{16+6/u^2}} = \frac{8}{-\sqrt{16}} = -2,$$

it follows that the graph of $y = \dfrac{8u-3}{\sqrt{16u^2+6}}$ has horizontal asymptotes of $y = \pm 2$.

61. $f(u) = \dfrac{2u^2-1}{\sqrt{6+u^4}}$

SOLUTION Because

$$\lim_{u\to\infty} \frac{2u^2-1}{\sqrt{6+u^4}} = \lim_{u\to\infty} \frac{2-1/u^2}{\sqrt{6/u^4+1}} = \frac{2}{\sqrt{1}} = 2$$

and

$$\lim_{u\to-\infty} \frac{2u^2-1}{\sqrt{6+u^4}} = \lim_{u\to-\infty} \frac{2-1/u^2}{\sqrt{6/u^4+1}} = \frac{2}{\sqrt{1}} = 2,$$

it follows that the graph of $y = \dfrac{2u^2-1}{\sqrt{6+u^4}}$ has a horizontal asymptote of $y = 2$.

62. $f(x) = \dfrac{3x^{2/3}+9x^{3/7}}{7x^{4/5}-4x^{-1/3}}$

SOLUTION Because

$$\lim_{x\to\infty} \frac{3x^{2/3}+9x^{3/7}}{7x^{4/5}-4x^{-1/3}} = \lim_{x\to\infty} \frac{3x^{-2/15}+9x^{-13/35}}{7-x^{-17/15}} = 0$$

and

$$\lim_{x\to-\infty} \frac{3x^{2/3}+9x^{3/7}}{7x^{4/5}-4x^{-1/3}} = \lim_{x\to-\infty} \frac{3x^{-2/15}+9x^{-13/35}}{7-x^{-17/15}} = 0,$$

it follows that the graph of $y = \dfrac{3x^{2/3}+9x^{3/7}}{7x^{4/5}-4x^{-1/3}}$ has a horizontal asymptote of $y = 0$.

63. $f(t) = \dfrac{t^{1/3}-t^{-1/3}}{(t-t^{-1})^{1/3}}$

SOLUTION Because

$$\lim_{t\to\infty} \frac{t^{1/3}-t^{-1/3}}{(t-t^{-1})^{1/3}} = \lim_{t\to\infty} \frac{1-t^{-2/3}}{(1-t^{-2})^{1/3}} = \frac{1}{1^{1/3}} = 1$$

and

$$\lim_{t\to-\infty} \frac{t^{1/3}-t^{-1/3}}{(t-t^{-1})^{1/3}} = \lim_{t\to-\infty} \frac{1-t^{-2/3}}{(1-t^{-2})^{1/3}} = \frac{1}{1^{1/3}} = 1,$$

it follows that the graph of $y = \dfrac{t^{1/3}-t^{-1/3}}{(t-t^{-1})^{1/3}}$ has a horizontal asymptote of $y = 1$.

64. Calculate (a)–(d), assuming that

$$\lim_{x\to3} f(x) = 6, \qquad \lim_{x\to3} g(x) = 4$$

(a) $\displaystyle\lim_{x\to3} (f(x) - 2g(x))$

(b) $\displaystyle\lim_{x\to3} x^2 f(x)$

(c) $\displaystyle\lim_{x\to3} \frac{f(x)}{g(x)+x}$

(d) $\displaystyle\lim_{x\to3} (2g(x)^3 - g(x)^{3/2})$

SOLUTION

(a) $\lim_{x\to 3}(f(x) - 2g(x)) = \lim_{x\to 3} f(x) - 2\lim_{x\to 3} g(x) = 6 - 2(4) = -2.$

(b) $\lim_{x\to 3} x^2 f(x) = \lim_{x\to 3} x^2 \cdot \lim_{x\to 3} f(x) = 3^2 \cdot 6 = 54.$

(c) $\lim_{x\to 3} \dfrac{f(x)}{g(x)+x} = \dfrac{\lim_{x\to 3} f(x)}{\lim_{x\to 3}(g(x)+x)} = \dfrac{6}{\lim_{x\to 3} g(x) + \lim_{x\to 3} x} = \dfrac{6}{4+3} = \dfrac{6}{7}.$

(d) $\lim_{x\to 3}(2g(x)^3 - g(x)^{3/2}) = 2\left(\lim_{x\to 3} g(x)\right)^3 - \left(\lim_{x\to 3} g(x)\right)^{3/2} = 2(4)^3 - 4^{3/2} = 120.$

65. Assume that the following limits exist:

$$A = \lim_{x\to a} f(x), \qquad B = \lim_{x\to a} g(x), \qquad L = \lim_{x\to a} \frac{f(x)}{g(x)}$$

Prove that if $L = 1$, then $A = B$. *Hint:* You cannot use the Quotient Law if $B = 0$, so apply the Product Law to L and B instead.

SOLUTION Suppose the limits A, B, and L all exist and $L = 1$. Then

$$B = B\cdot 1 = B\cdot L = \lim_{x\to a} g(x) \cdot \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} g(x)\frac{f(x)}{g(x)} = \lim_{x\to a} f(x) = A.$$

66. GU Define $g(t) = (1 + 2^{1/t})^{-1}$ for $t \neq 0$. How should $g(0)$ be defined to make $g(t)$ left-continuous at $t = 0$?

SOLUTION Because

$$\lim_{t\to 0-}(1 + 2^{1/t})^{-1} = \left[\lim_{t\to 0-}(1 + 2^{1/t})\right]^{-1} = 1^{-1} = 1,$$

we should define $g(0) = 1$ to make $g(t)$ left-continuous at $t = 0$.

67. In the notation of Exercise 65, give an example where L exists but neither A nor B exists.

SOLUTION Suppose

$$f(x) = \frac{1}{(x-a)^3} \qquad \text{and} \qquad g(x) = \frac{1}{(x-a)^5}.$$

Then, neither A nor B exists, but

$$L = \lim_{x\to a} \frac{(x-a)^{-3}}{(x-a)^{-5}} = \lim_{x\to a}(x-a)^2 = 0.$$

68. True or false?

(a) If $\lim_{x\to 3} f(x)$ exists, then $\lim_{x\to 3} f(x) = f(3)$.

(b) If $\lim_{x\to 0} \dfrac{f(x)}{x} = 1$, then $f(0) = 0$.

(c) If $\lim_{x\to -7} f(x) = 8$, then $\lim_{x\to -7} \dfrac{1}{f(x)} = \dfrac{1}{8}$.

(d) If $\lim_{x\to 5+} f(x) = 4$ and $\lim_{x\to 5-} f(x) = 8$, then $\lim_{x\to 5} f(x) = 6$.

(e) If $\lim_{x\to 0} \dfrac{f(x)}{x} = 1$, then $\lim_{x\to 0} f(x) = 0$.

(f) If $\lim_{x\to 5} f(x) = 2$, then $\lim_{x\to 5} f(x)^3 = 8$.

SOLUTION

(a) False. The limit $\lim_{x\to 3} f(x)$ may exist and need not equal $f(3)$. The limit is equal to $f(3)$ if $f(x)$ is continuous at $x = 3$.

(b) False. The value of the limit $\lim_{x\to 0} \dfrac{f(x)}{x} = 1$ does not depend on the value $f(0)$, so $f(0)$ can have any value.

(c) True, by the Limit Laws.

(d) False. If the two one-sided limits are not equal, then the two-sided limit does not exist.

(e) True. Apply the Product Law to the functions $\dfrac{f(x)}{x}$ and x.

(f) True, by the Limit Laws.

69. Let $f(x) = x\left[\frac{1}{x}\right]$, where $[x]$ is the greatest integer function. Show that for $x \neq 0$,

$$\frac{1}{x} - 1 < \left[\frac{1}{x}\right] \le \frac{1}{x}$$

Then use the Squeeze Theorem to prove that

$$\lim_{x\to 0} x\left[\frac{1}{x}\right] = 1$$

Hint: Treat the one-sided limits separately.

SOLUTION Let y be any real number. From the definition of the greatest integer function, it follows that $y - 1 < [y] \le y$, with equality holding if and only if y is an integer. If $x \neq 0$, then $\frac{1}{x}$ is a real number, so

$$\frac{1}{x} - 1 < \left[\frac{1}{x}\right] \le \frac{1}{x}.$$

Upon multiplying this inequality through by x, we find

$$1 - x < x\left[\frac{1}{x}\right] \le 1.$$

Because

$$\lim_{x\to 0}(1 - x) = \lim_{x\to 0} 1 = 1,$$

it follows from the Squeeze Theorem that

$$\lim_{x\to 0} x\left[\frac{1}{x}\right] = 1.$$

70. Let r_1 and r_2 be the roots of $f(x) = ax^2 - 2x + 20$. Observe that $f(x)$ "approaches" the linear function $L(x) = -2x + 20$ as $a \to 0$. Because $r = 10$ is the unique root of $L(x)$, we might expect one of the roots of $f(x)$ to approach 10 as $a \to 0$ (Figure 3). Prove that the roots can be labeled so that $\lim_{a\to 0} r_1 = 10$ and $\lim_{a\to 0} r_2 = \infty$.

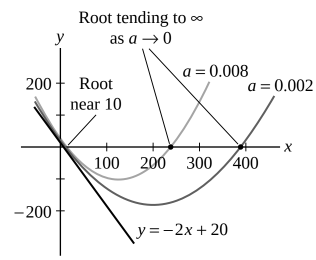


FIGURE 3 Graphs of $f(x) = ax^2 - 2x + 20$.

SOLUTION Using the quadratic formula, we find that the roots of the quadratic polynomial $f(x) = ax^2 - 2x + 20$ are

$$\frac{2 \pm \sqrt{4 - 80a}}{2a} = \frac{1 \pm \sqrt{1 - 20a}}{a} = \frac{20}{1 \pm \sqrt{1 - 20a}}.$$

Now let

$$r_1 = \frac{20}{1 + \sqrt{1 - 20a}} \qquad \text{and} \qquad r_2 = \frac{20}{1 - \sqrt{1 - 20a}}.$$

It is straightforward to calculate that

$$\lim_{a\to 0} r_1 = \lim_{a\to 0} \frac{20}{1 + \sqrt{1 - 20a}} = \frac{20}{2} = 10$$

and that

$$\lim_{a\to 0} r_2 = \lim_{a\to 0} \frac{20}{1 - \sqrt{1 - 20a}} = \infty$$

as desired.

71. Use the IVT to prove that the curves $y = x^2$ and $y = \cos x$ intersect.

SOLUTION Let $f(x) = x^2 - \cos x$. Note that any root of $f(x)$ corresponds to a point of intersection between the curves $y = x^2$ and $y = \cos x$. Now, $f(x)$ is continuous over the interval $[0, \frac{\pi}{2}]$, $f(0) = -1 < 0$ and $f(\frac{\pi}{2}) = \frac{\pi^2}{4} > 0$. Therefore, by the Intermediate Value Theorem, there exists a $c \in (0, \frac{\pi}{2})$ such that $f(c) = 0$; consequently, the curves $y = x^2$ and $y = \cos x$ intersect.

72. Use the IVT to prove that $f(x) = x^3 - \dfrac{x^2 + 2}{\cos x + 2}$ has a root in the interval $[0, 2]$.

SOLUTION Let $f(x) = x^3 - \frac{x^2+2}{\cos x+2}$. Because $\cos x + 2$ is never zero, $f(x)$ is continuous for all real numbers. Because

$$f(0) = -\frac{2}{3} < 0 \qquad \text{and} \qquad f(2) = 8 - \frac{6}{\cos 2 + 2} \approx 4.21 > 0,$$

the Intermediate Value Theorem guarantees there exists a $c \in (0, 2)$ such that $f(c) = 0$.

73. Use the IVT to show that $e^{-x^2} = x$ has a solution on $(0, 1)$.

SOLUTION Let $f(x) = e^{-x^2} - x$. Observe that f is continuous on $[0, 1]$ with $f(0) = e^0 - 0 = 1 > 0$ and $f(1) = e^{-1} - 1 < 0$. Therefore, the IVT guarantees there exists a $c \in (0, 1)$ such that $f(c) = e^{-c^2} - c = 0$.

74. Use the Bisection Method to locate a solution of $x^2 - 7 = 0$ to two decimal places.

SOLUTION Let $f(x) = x^2 - 7$. By trial and error, we find that $f(2.6) = -0.24 < 0$ and $f(2.7) = 0.29 > 0$. Because $f(x)$ is continuous on $[2.6, 2.7]$, it follows from the Intermediate Value Theorem that $f(x)$ has a root on $(2.6, 2.7)$. We approximate the root by the midpoint of the interval: $x = 2.65$. Now, $f(2.65) = 0.0225 > 0$. Because $f(2.6)$ and $f(2.65)$ are of opposite sign, the root must lie on $(2.6, 2.65)$. The midpoint of this interval is $x = 2.625$ and $f(2.625) < 0$; hence, the root must be on the interval $(2.625, 2.65)$. Continuing in this fashion, we construct the following sequence of intervals and midpoints.

interval	midpoint
$(2.625, 2.65)$	2.6375
$(2.6375, 2.65)$	2.64375
$(2.64375, 2.65)$	2.646875
$(2.64375, 2.646875)$	2.6453125
$(2.6453125, 2.646875)$	2.64609375

At this point, we note that, to two decimal places, one root of $x^2 - 7 = 0$ is 2.65.

75. Give an example of a (discontinuous) function that does not satisfy the conclusion of the IVT on $[-1, 1]$. Then show that the function

$$f(x) = \begin{cases} \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

satisfies the conclusion of the IVT on every interval $[-a, a]$, even though f is discontinuous at $x = 0$.

SOLUTION Let $g(x) = [x]$. This function is discontinuous on $[-1, 1]$ with $g(-1) = -1$ and $g(1) = 1$. For all $c \neq 0$, there is no x such that $g(x) = c$; thus, $g(x)$ does not satisfy the conclusion of the Intermediate Value Theorem on $[-1, 1]$.

Now, let

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

and let $a > 0$. On the interval

$$x \in \left[\frac{a}{2 + 2\pi a}, \frac{a}{2}\right] \subset [-a, a],$$

$\frac{1}{x}$ runs from $\frac{2}{a}$ to $\frac{2}{a} + 2\pi$, so the sine function covers one full period and clearly takes on every value from $-\sin a$ through $\sin a$.

76. Let $f(x) = \dfrac{1}{x+2}$.

(a) Show that $\left|f(x) - \frac{1}{4}\right| < \dfrac{|x-2|}{12}$ if $|x - 2| < 1$. *Hint:* Observe that $|4(x + 2)| > 12$ if $|x - 2| < 1$.

(b) Find $\delta > 0$ such that $\left|f(x) - \frac{1}{4}\right| < 0.01$ for $|x - 2| < \delta$.

(c) Prove rigorously that $\lim\limits_{x\to 2} f(x) = \frac{1}{4}$.

SOLUTION

(a) Let $f(x) = \frac{1}{x+2}$. Then

$$\left|f(x) - \frac{1}{4}\right| = \left|\frac{1}{x+2} - \frac{1}{4}\right| = \left|\frac{4 - (x+2)}{4(x+2)}\right| = \frac{|x-2|}{|4(x+2)|}.$$

If $|x - 2| < 1$, then $1 < x < 3$, so $3 < x + 2 < 5$ and $12 < 4(x + 2) < 20$. Hence,

$$\frac{1}{|4(x+2)|} < \frac{1}{12} \qquad \text{and} \qquad \left|f(x) - \frac{1}{4}\right| < \frac{|x-2|}{12}.$$

(b) If $|x - 2| < \delta$, then by part (a),

$$\left|f(x) - \frac{1}{4}\right| < \frac{\delta}{12}.$$

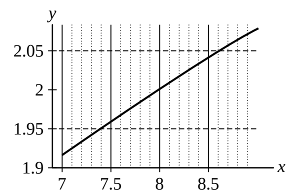
Choosing $\delta = 0.12$ will then guarantee that $|f(x) - \frac{1}{4}| < 0.01$.

(c) Let $\epsilon > 0$ and take $\delta = \min\{1, 12\epsilon\}$. Then, whenever $|x - 2| < \delta$,

$$\left| f(x) - \frac{1}{4} \right| = \left| \frac{1}{x+2} - \frac{1}{4} \right| = \frac{|2 - x|}{4|x+2|} \leq \frac{|x-2|}{12} < \frac{\delta}{12} = \epsilon.$$

77. GU Plot the function $f(x) = x^{1/3}$. Use the zoom feature to find a $\delta > 0$ such that $|x^{1/3} - 2| < 0.05$ for $|x - 8| < \delta$.

SOLUTION The graphs of $y = f(x) = x^{1/3}$ and the horizontal lines $y = 1.95$ and $y = 2.05$ are shown below. From this plot, we see that $\delta = 0.55$ guarantees that $|x^{1/3} - 2| < 0.05$ whenever $|x - 8| < \delta$.

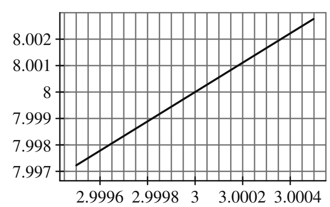


78. Use the fact that $f(x) = 2^x$ is increasing to find a value of δ such that $|2^x - 8| < 0.001$ if $|x - 2| < \delta$. *Hint:* Find c_1 and c_2 such that $7.999 < f(c_1) < f(c_2) < 8.001$.

SOLUTION From the graph below, we see that

$$7.999 < f(2.99985) < f(3.00015) < 8.001.$$

Thus, with $\delta = 0.00015$, it follows that $|2^x - 8| < 0.001$ if $|x - 3| < \delta$.



79. Prove rigorously that $\lim_{x \to -1} (4 + 8x) = -4$.

SOLUTION Let $\epsilon > 0$ and take $\delta = \epsilon/8$. Then, whenever $|x - (-1)| = |x + 1| < \delta$,

$$|f(x) - (-4)| = |4 + 8x + 4| = 8|x + 1| < 8\delta = \epsilon.$$

80. Prove rigorously that $\lim_{x \to 3} (x^2 - x) = 6$.

SOLUTION Let $\epsilon > 0$ and take $\delta = \min\{1, \epsilon/6\}$. Because $\delta \leq 1$, $|x - 3| < \delta$ guarantees $|x + 2| < 6$. Therefore, whenever $|x - 3| < \delta$,

$$|f(x) - 6| = |x^2 - x - 6| = |x - 3|\,|x + 2| < 6|x - 3| < 6\delta \leq \epsilon.$$

Chapter 2: Limits
Preparing for the AP Exam Solutions

Multiple Choice Questions

1) B	2) A	3) C	4) D	5) E	6) B
7) E	8) B	9) C	10) A	11) C	12) A
13) D	14) C	15) B	16) B	17) D	18) E
19) E	20) B				

Free Response Questions

1. a) $\dfrac{f(\frac{3\pi}{2})-f(\frac{\pi}{2})}{\frac{3\pi}{2}-\frac{\pi}{2}}=\dfrac{\frac{-1}{(\frac{3\pi}{2})}-\frac{1}{(\frac{\pi}{2})}}{\pi}=\dfrac{-1}{\pi}(\dfrac{2}{3\pi}+\dfrac{2}{\pi})=\dfrac{-8}{3\pi^2}$

b) $\lim_{x\to 0} f(x)=1$

c) No, $\lim_{x\to 0} f(x)=1$, so neither the left-hand limit nor the right hand limit is infinite, which is needed for the graph to have a vertical asymptote.

d) We know $-1\le \sin x\le 1$, so if $x>0$, then $\dfrac{-1}{x}\le\dfrac{\sin x}{x}\le\dfrac{1}{x}$, and since $\lim_{x\to\infty}\dfrac{-1}{x}=0=\lim_{x\to\infty}\dfrac{1}{x}$, the Squeeze Theorem implies $\lim_{x\to\infty}\dfrac{\sin x}{x}=0$. This means the line $y = 0$ is a horizontal asymptote.

POINTS:
(a) (2 pts) 1) change in y; 1) answer
(b) (1 pt)
(c) (3 pts) 1) "no"; 1) mentioning finite limit; 1) mentioning need for infinite limit
(d) (3 pts) 1) $\dfrac{-1}{x}\le\dfrac{\sin x}{x}\le\dfrac{1}{x}$; 1) $\lim_{x\to\infty}\dfrac{-1}{x}=0=\lim_{x\to\infty}\dfrac{1}{x}$; 1) conclusion

2. a) The function $f(x)=\dfrac{x^2-7x+10}{x^2-25}$ is discontinuous at $x = 5$ and $x = -5$. First, $\lim_{x\to 5}\dfrac{x^2-7x+10}{x^2-25}$ = $\lim_{x\to 5}\dfrac{(x-5)(x-2)}{(x-5)(x+5)}$ = $\lim_{x\to 5}\dfrac{(x-2)}{(x+5)}$ = $\dfrac{3}{10}$. Thus the line $x = 5$ is not a vertical asymptote. Next, $\lim_{x\to -5^+}\dfrac{x^2-7x+10}{x^2-25}$ = $\lim_{x\to -5^+}\dfrac{x-2}{x+5}$ = $-\infty$ Thus the line $x = -5$ is a vertical asymptote.

b) $\lim_{x\to\infty}\dfrac{x^2-7x+10}{x^2-25}$ = 1, so the line $y = 1$ is a horizontal asymptote. Also $\lim_{x\to -\infty}\dfrac{x^2-7x+10}{x^2-25}$ = 1, so the line $y = 1$ is the only horizontal asymptote.

c) Yes, since $\lim_{x\to 5} f(x)=\dfrac{3}{10}$, we can let $A = \dfrac{3}{10}$.

d) No, since $\lim_{x\to -5} f(x)$ does not exist, there is no possible value for *B*.

POINTS:

(a) (4 pts) 1) "no" for $x = 5$; 1) Limit is $\frac{3}{10}$; 1) "yes" for $x = -5$; 1) infinite limit

(b) (3 pts) 1) $y = 1$; 1) Limit at ∞. 1) Limit at $-\infty$.

(c) (1pt) $A = \frac{3}{10}$

(d) (1 pt) No limit.

3. a) Since $-5 \le f(x) \le 10$, if $x > 0$ then $-5x \le xf(x) \le 10x$. Thus by the Squeeze Theorem $\lim_{x\to 0^+} xf(x) =$ 0. Next, if $x < 0$, then $-5x \ge xf(x) \ge 10x$. Applying the Squeeze Theorem again, $\lim_{x\to 0^-} xf(x) = 0$. Thus $\lim_{x\to 0} xf(x) = \lim_{x\to 0} g(x) = 0$. Checking the functional value, we have $g(0) = 0 \cdot 3 = 0$. Thus $\lim_{x\to 0} g(x) = g(0)$, so *g* is continuous at $x = 0$.

b) No. $\lim_{x\to 0} \frac{g(x)-0}{x-0} = \lim_{x\to 0} \frac{xf(x)}{x} = \lim_{x\to 0} f(x)$, which does not exist.

POINTS:

(a) (6 pts) 1) $g(0) = 0$; 1) if $x > 0$ then $-5x \le xf(x) \le 10x$; 1) $\lim_{x\to 0^+} xf(x) = 0$;
if $x < 0$, then $-5x \ge xf(x) \ge 10x$; 1) $\lim_{x\to 0^-} xf(x) = 0$; 1) $\lim_{x\to 0} g(x) = 0$

(b) (3 pts) 1) Considers $\lim_{x\to 0} \frac{g(x)-0}{x-0}$; 1) $\lim_{x\to 0} \frac{g(x)-0}{x-0} = \lim_{x\to 0} f(x)$; 1) Answer

4. a) First, $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (6-x) = 2$. Next $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \sqrt[3]{2x} = \sqrt[3]{8} = 2$.

So $\lim_{x\to 4} f(x) = 2$. Also *f*(4) = 2, which means *f* is continuous at $x = 4$.

b) $\frac{f(.004) - f(0)}{.004 - 0} = \frac{\sqrt[3]{.008}}{.004} = \frac{.2}{.004} = 50$

c) No, $\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\sqrt[3]{2x}}{x} = \lim_{x\to 0} \frac{\sqrt[3]{2}}{x^{2/3}}$ does not exist.

POINTS:

(a) (5 pts) 1) $\lim_{x\to 4^+} (6-x) = 2$; 1) $\lim_{x\to 4^-} \sqrt[3]{2x} = 2$; 1) $\lim_{x\to 4} f(x) = 2$; 1) *f*(4) = 2; 1) Answer

(b) (1 pt)

(c) (3 pts) 1) Considers $\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$; 1) $\lim_{x\to 0} \frac{\sqrt[3]{2x}}{x}$; 1) $\lim_{x\to 0} \frac{\sqrt[3]{2}}{x^{2/3}}$ does not exist.